STUDY ON FUZZY MATHEMATICS

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Supervisor

Dr.J.S.Parihar
Reader,
Deptt. of Maths.
Pt.J.N.P.G.College,
Banda (U.P.)

By

Santosh Kumar Singh Bhadauria
M.Sc.
Research Scholar
Deptt. of Maths.
Pt. J. N. P. G. College,
Banda (U.P.)
Enrolment No. BU/96/523

Certificate

I hereby certify that Santosh Kumar Singh Bhadauria's thesis titled, "Study On Fuzzy Mathematics", Submitted to the Bundelkhand University, Jhansi for the award of Ph.D. degree is a bonafied research work carried on under my supervision and that the candidate has worked for the prescribed period of the university ordinance.

His research work is original and to the best of my knowledge it has not been submitted elsewhere for the award of any degree or diploma in present form.

Dr.J.S.Parihar

(Supervisor)
Reader, Deptt. of Maths.,
Pt.J.N.P.G.College, Banda (U.P.)

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Santosh Singh

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Preface

During the last thirty five years some mathematicians have engaged their attention to a new branch of mathematics which is called "Fuzzy Mathematics". "Perfect notions" or "exact concepts" are found in pure mathematics. But in real life "inexact structures" are found. To meet the problems dealing with fuzzy (uncertain) situations, fuzzy set theory has been developed by L.A.Zadeh in 1965. Subsequently several new branches such as Fuzzy algebra, Fuzzy vector space, Fuzzy group, Fuzzy topology, Fuzzy analysis etc. are being developed. The present thesis is a part of study of these developing new branches of mathematics.

This thesis consists of seven chapters.

In Chapter - I, general leterature and information about fuzzy mathematics have been given. In this chapter, the structures of fuzzy sets with respect to algebraic operations of union, intersection and complement has been studied.

In Chapter - II, a study of various kind of fuzzy relations have been done for fuzzy binary relations. A generalization to n' ary relations is a straight forward.

In Chapter - III, the concept of fuzzy sub group, introduced by A. Rosenfeld has been analysed under different conditions. The concept of fuzzy normal sub group has been introduced and several theorems have been presented.

In Chapter IV, Fuzzy Vector space and Fuzzy vector sub space have been defined, and several properties of vector space have been examined in fuzzy vector space.

In Chapter - V, Fuzzy topology has been presented and results of topology in fuzzy topology have been examined.

A study of fuzzy inegration and fuzzy differentation is presented in Chapter VI. The concepts have been cleared with suitable examples and graphs.

Chapter - VII is devoted to a comparative study of fuzzy and probabilistic measures of informations.

Santosh Singh.

(Santosh Kumar Singh Bhadauria)

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CONTENTS

Chapters :

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CHAPTER - I

General Introduction

Introduction :

Fuzzy Set Theory -

During the last thirty five year's some mathematicians have engaged their attention to a new branch of mathematics which is called Fuzzy Mathematics. "Perfect notions" or "exact concepts" are found in pure mathematics. But in real life "inexact structures" are found. To meet the prolems dealing with Fuzzy (uncertain) situations, Fuzzy set theory has been developed by L.A.Zadeh in 1965. Subsequently several new branches of mathematics such as Fuzzy Algebra, Fuzzy vector space, Fuzzy group, Fuzzy Topology are being developed.

Basic Definition's -

1.1 - Classical (crisp) set- It is defined as a collection of elements or objects $x \in X$ which can be finite, countable or uncountable. Each single element can either belong to or not belog to a set A, $A \subseteq X$.

1.2 - Fuzzy Set : If X is a collection of objects denoted by x then a fuzzy set $\tilde{A}$ in X is a set of ordered pairs :

$$\tilde{A} = \{ (x, \mu_{\tilde{A}}(x)) : x \in X \}$$

$\mu_{\tilde{A}}(x)$ is called the membership function or grade of membership of x is $\tilde{A}$

Note - If $\mu_{\tilde{A}} (x_i)) = 0$ or $1, \forall i$, then $\tilde{A}$ is non fuzzy set i.e. classical or crisp set. Thus fuzzy set is a generalization of crisp set. The elements with a zero degree of membership are normally not listed. The range of the membership function is a subset of non-negative real numbers whose supremum is finite.

Example 1.1

The manager of a lodge want's to classify the house he offer's to his client's. One indicator of comfort of these houses is the number of bedrooms in it. Let $X = \{ 1,2,3,4,............, 10 \}$ be the set of available types of houses described by x = number of bedrooms in a house. Then the fuzzy set "comfortable type of house for a 4 person family may be described as

$\tilde{A} = \{ (1, .2), (2, .5), (3, .8), (4,1), (5, .7), (6, .3) \}$

Note : A fuzzy set is denoted by an ordered set of pairs, The first element of which denotes the element and the second the degree of membership.

Ex.1.2 $\tilde{A}$ = "real number's considerably larger than 10"

or $\tilde{A} = \{ (x, \mu_{\tilde{A}} (x)) : x \in X \}$

Where $\mu_{\tilde{A}}(x) = \begin{cases} 0 , & x \leq 10 \\ \{1+(x-10)^{-2} \}^{-1}, & x > 10. \end{cases}$

Ex. 1.3 $\tilde{A}$ = "real numbers close to 10"

or $\tilde{A} = \{ (x , \mu_{\tilde{A}}(x)) \text{ where}$

$$\mu_{\tilde{A}}(x) = \{ 1 + (x - 10)^2 \}^{-1} \}$$

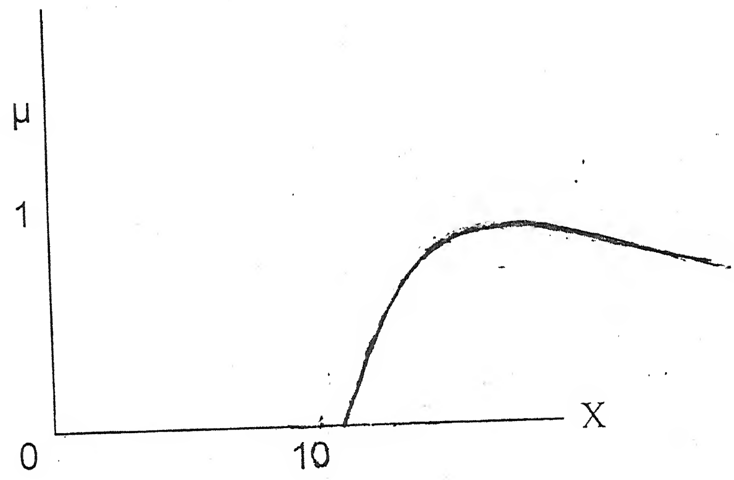


Fig. 1.1

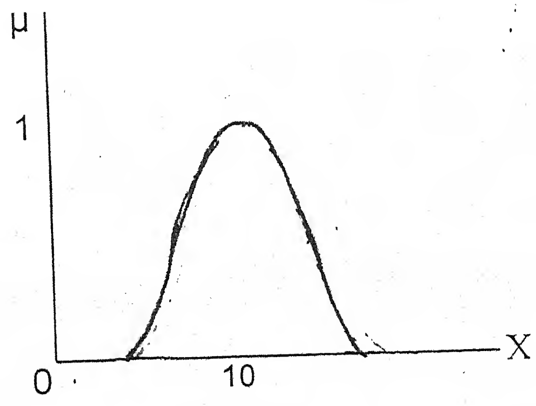


Fig. 1.2

Definition -

1.3 The support S of a fuzzy set $\tilde{A}$, is defined as the crisp set

of all $x \in X$ such that $\mu_{\tilde{A}}(x) > 0$.

Ex. 1.4 In example 1.1; $X = \{1,2,3,........10\}$ and

$\tilde{A} = \{ (1,.2), (2.5), (3,.8), (4.1), (5,.7), (6,.3) \}$

$S(\tilde{A}) = \{ 1, 2, 3, 4, 5, 6 \}$

Definition -

1.4 - α Level Set

The crisp set of elements that belong to the fuzzy set

$\tilde{A}$ atlest to the degree α is called the α level set.

i.e. $A_\alpha = \{ x \in X : \mu_{\tilde{A}}(x) > \alpha \}$

also $A'_\alpha = \{ x \in X : \mu_{\tilde{A}}(x) > \alpha \}$ is called strong α level set.

or strong α - cut.

(Ex.1.5) - In previous example (1.1), $X = \{1,2,3,...........10\}$

$\tilde{A} = \{ (1, .2), (2, .5), (3, .8), (4,1), (5, .7), (6,.3) \}$

The α - level set's are

$A_{.2} = \{ 1, 2, 3, 4, 5, 6 \}$

$A_{.3} = \{ 2, 3, 4, 5, 6 \}$

$A_{.5} = \{ 2, 3, 4, 5 \}$

$A_{.7} = \{ 3, 4, 5 \}$

$A_{.8} = \{ 3, 4 \}$

$A_{1} = \{ 4 \}$

$A_{.4} = \{ 2, 3, 4, 5 \}$

and $A'_{.3} = \{ 2, 3, 4, 5 \}$

Definition

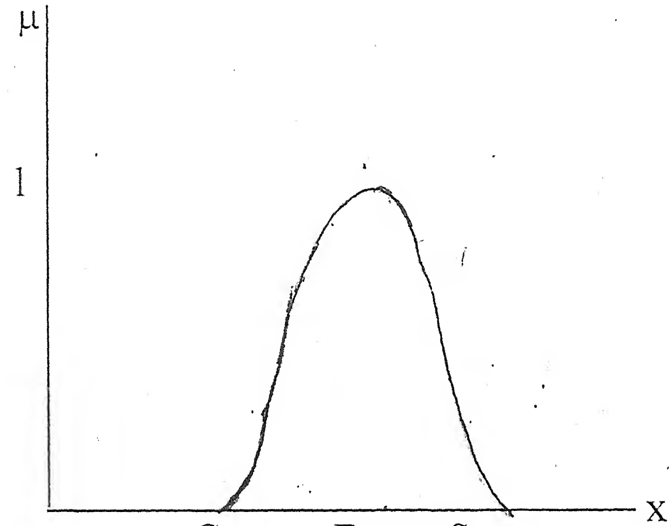
1.5 A fuzzy set $\tilde{A}$ is convex if

$\mu_{\tilde{A}} (\lambda x_1 + (1- \lambda) x_2 \} \geq \min. \{ \mu_{\tilde{A}} (x_1), \mu_{\tilde{A}} (x_2); \}$

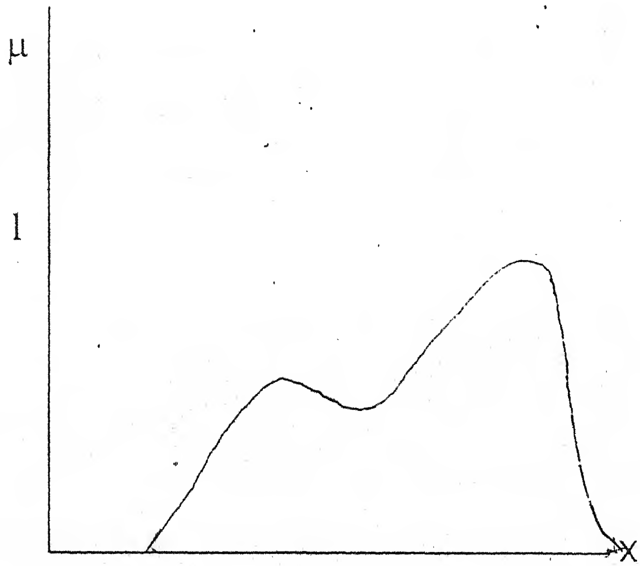
$x_{1,}\ x_2 \in X , \lambda \in [0, 1]$

Alternatively, A Fuzzy set is convex if all α – level sets

are convex



Convex Fuzzy Set
Fig. 1.3



Non convex Fuzzy Set
Fig. 1.4

Here Fig. 1.3 shows a convex fuzzy set and fig. 1.4 shows a non convex fuzzy set.

Definition 1.6 - The cardinality of a fuzzy set

It is defined by $|\tilde{A}| = \sum_{x \in X} \mu_{\tilde{A}}(x)$

The relative cardinality of fuzzy set $\tilde{A}$ is defined by

$$||\tilde{A}|| = \frac{|\tilde{A}|}{|X|}$$

Example 1.6 - In previous example 1.1, X = { 1, 2, 3, ...10 }

$\tilde{A}$ = { (1..2), (2.5), (3,.8), (4,1), (5,.7), (6, .3) }

$|\tilde{A}| = .2 + .5 + .8 + 1 + .7 + .3 = 3.5$

It's relative cardinality is

$$||\tilde{A}|| = \frac{3.5}{10} = 0.35$$

Operations on Fuzzy sets

<u>Algebraic operations</u> :-

1.7 Def. : Cartesian product of fuzzy sets :-

Let $\tilde{A}_1$ and $\tilde{A}_2$ be fuzzy sets in X_1 and X_2. The

cartesian product is then a fuzzy set in the product space $X_1 \times X_2$ with the membership function

$$\mu(\tilde{A}_1 \times \tilde{A}_2)(x) = \min.\{\mu_{\tilde{A}_i}(x_i)\} : x_i \in X_i :$$

1.8 Defs. - The m th power of a fuzzy set Ã is a fuzzy set with the membership function -

$$\mu(\tilde{A}^m(x) = [\mu\tilde{A}(x)]^m, x \in X$$

1.9 Def. - The algebraic sum (Probabilistic sum)

$\tilde{C} = \tilde{A} + \tilde{B}$ is defined by

$$\tilde{C} = \{x, \mu_{(\tilde{A}+\tilde{B})}(x)\}, x \in X$$

Where $\mu_{(\tilde{A}+\tilde{B})}(x) = \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)$

1.10 Def. - The bounded sum $\tilde{C} = \tilde{A} \oplus \tilde{B}$ is defined by

$$\tilde{C} = \{x, (\mu_{(\tilde{A}\oplus\tilde{B})}(x)\} : x \in X$$

Where $\mu_{(\tilde{A}\oplus\tilde{B})}(x) = \min\{1, \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x)\}$

1.11 Def. - The bounded difference $C = \tilde{A} \ominus \tilde{B}$ is defined by

$$\tilde{C} = \{x, \mu_{(\tilde{A}\ominus\tilde{B})}(x)\} : x \in X$$

Where $\mu_{(\tilde{A} \ominus \tilde{B})}(x) = \max\{0, \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x)\} - 1\}$

1.12 Def. - The algebraic product of two fuzzy sets $\tilde{C} = \tilde{A}.\tilde{B}$ is

defined by

$$\tilde{C} = \{ x, \mu_{\tilde{A}}(x) \,.\, \mu_{\tilde{B}}(x) \} : x \in X$$

1.7 Example :

Let $\tilde{A}(x) = \{3,.5), (5,1), (7,.6)\}$

$\tilde{B}(x) = \{3,1), (5, .6)\}$

Then $\tilde{A} \times \tilde{B} = [\{(3,3), .5\}. \{(5,3), 1\}. \{(7,3), .6\}, \{(3,5), .5\},$

$\{ (5, 5), .6 \}, \{(7,5), .6\}]$

$\tilde{A} = \{(3, .5), (5,1), (7, .6) \}$

$\tilde{B} = \{ (3,1), (5, .6) (7, 0)\}$

$\tilde{A}^2 = \{ (3, .25), (5,1), (7, .36) \}$

$\tilde{A} + \tilde{B} = \{ (3,1), (5,1), (7, .6) \}$

$\tilde{A} \oplus \tilde{B} = \{ (3,1), (5,1), (7, .6) \}$

$\tilde{A} \ominus \tilde{B} = \{ (3,.5), (5, .6), (7, 0) \}$

$= \{ (3,.5), (5, .6) \}$

$P (A + B) = P (A) + P (B) - P (A) P (B)$

$= 5 + 1 - .5 X 1$

$= 1$

$\tilde{A} . \tilde{B} \quad = \{ (3, .5), (5, 0.6), (7, 0) \}$

$= \{ (3, .5), (5, .6) \}$

1.13 Def. The membership function $\mu_{\tilde{C}}(x)$ of the intersection

$\tilde{C} = \tilde{A} \cap \tilde{B}$ is pointwise defined by

$\mu_{\tilde{C}}(x) = \min. \{ \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) \}, \ x \in X$

1.14 Def. The membership function $\mu_{\tilde{D}}(x)$ of union

$\tilde{D} = \tilde{A} \cap \tilde{B}$ is pointwise defined by

$\mu_{\tilde{D}}(x) = \max. \{ \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) \}, \ x \in X$

1.15 Def. The membership function of the complement of a fuzzy set $\tilde{A}$, $\mu_{\tilde{A}}(x)$ is defined by

$\mu_{\not\subset \tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x) , \ x \in X ,$

1.8 Example :-

Let $\widetilde{A}$ be the fuzzy set defined by

$\widetilde{A}$ = { (1, .2), (2, .5), (3, .8), (4, 1), (5, .7), (6, .3) }

& $\widetilde{B}$ = { (3, .2), (4, .4), (5, .6), (6, .8), (7, 1), (8, 1) }

Then $\widetilde{A} \cap \widetilde{B}$ = { (3, .2), (4, .4), (5, .6), (6, .3) }

$\widetilde{A} \cup \widetilde{B}$ = { (1, .2), (2, .5), (3, .8), (4, 1), (5, .7), (6, .8),

(7, 1), (8, 1) }

The complement of $\widetilde{B}$ is

$\not\subset\widetilde{B}$ = { (1, 1), (2, 1), (3, .8), (4, .6), (5, .4), (6, .2) (7, 0),

(8, 0), (9, 1), (10, 1) }

For X = { 1, 2, 3, ...10 }

= { (1,1), (2,1), (3,.8), (4,.6), (5,.4), (6,.2), (9,1),(10,1) }

Def. : Equality of two fuzzy sets : If $\widetilde{A}$ and B are two fuzzy sets on X, then we say that

$\widetilde{A} = \widetilde{B}$ if $\mu_{\widetilde{A}}(x) = \mu_{\widetilde{B}}(x)$ for all $x \in X$

or $\tilde{A}(x) = \tilde{B}(x)$ for all $x \in X$

Def. : Fuzzy sub set of a fuzzy set :-

If $\tilde{A}$ and $\tilde{B}$ are fuzzy sets in X; we say $\tilde{A}$ is a sub set of $\tilde{B}$ ($\tilde{A} \subset \tilde{B}$) if and only if $\tilde{A}(x) \subseteq \tilde{B}(x)$ for all $x \in X$

Algebraic properties of union intersection and complement :

Let $\tilde{A}, \tilde{B}, \tilde{C}$ be fuzzy sets in X. The following laws of ordinary set theory hold in fuzzy set theory also.

(a) Commutative law :

$$\tilde{A} \cup \tilde{B} = \tilde{B} \cup \tilde{A},\ \tilde{A} \cap \tilde{B} = \tilde{B} \cap \tilde{A}.$$

(b) Associative law :

$$\tilde{A} \cup (\tilde{B} \cup \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cup \tilde{C}$$

(c) Idempotency law :-

$$\tilde{A} \cup \tilde{A} = \tilde{A},\ \tilde{A} \cap \tilde{A} = \tilde{A}$$

(d) Distributive law :-

(i) $\tilde{A} \cup (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cap (\tilde{A} \cup \tilde{C})$

(ii) $\tilde{A} \cap (\tilde{B} \cup \tilde{C}) = (\tilde{A} \cap \tilde{B}) \cup (\tilde{A} \cap \tilde{C})$

(e) Absorption law :

$$\tilde{A} \cup (\tilde{A} \cap \tilde{B}) = \tilde{A},\ \tilde{A} \cap (\tilde{A} \cup \tilde{B}) = \tilde{A}$$

(f) De Margan laws :

$$\overline{(\tilde{A} \cap \tilde{B})} = \overline{\tilde{A}} \cup \overline{\tilde{B}}$$

$$\overline{(\tilde{A} \cup \tilde{B})} = \overline{\tilde{A}} \cap \overline{\tilde{B}}$$

(f) Law of Involution :

$$\overline{\{(\overline{\tilde{A}})\}} = \tilde{A}.$$

(h) Identity law :

(i) $\tilde{A} \cup \phi = \tilde{A},\ \tilde{A} \cap X = \tilde{A}$

(ii) $\tilde{A} \cap \phi = \phi,\ \tilde{A} \cup X = X$

<u>Partial order Fuzzy Set</u> :

We define the relation " $\leq$ " on the set of all fuzzy sets, denoted by $\tilde{p}(x)$ on x as

A $\leq$ B if A C B, where A & B are fuzzy sets on X.

Then we find that ($\widetilde{P}$ (X), $\leq$) is partially ordered set :

If we define supremum and infirmum of a family of fuzzy sets on X as the union and the intersection of the family, we find that the set of all fuzzy sets on X forms a complete distributive lattice. However, unlike ordinary set theory, $\widetilde{P}$ (X) does not form a Boolean algebra because the two laws of complementation holding in set theory do not hold in fuzzy set theory namely the laws of complementation.

(i) $A \cup \overline{A} = X,$

(ii) $A \cap \overline{A} = \phi$

do not hold in fuzzy set theory.

Hence we call the algebraic structure of fuzzy sets on any universe X as pseudo complemented Lattice with

respect to operations of union, intersection and complement.

In the light of above structure of fuzzy set theory, we define fuzzy algebra as follows :-

Fuzzy algebra :

A fuzzy algebra is defined to be the system $S = \angle S, +, *, - >$ where S has atleast two distinct elements and for all x, y, z $\in$ S, system S satisfies the following axioms.

(1) Idempotency : $X + X = X$, $X * X = X$

(2) Commutativity :

$X + Y = Y + X$, $X * Y = Y * X$

(3) Associativity :

$(X + Y) + Z = X + (Y + Z)$

and $(X * Y) * Z = X * (Y * Z)$

(4) Absorption :

$$X + (X * Y) = X$$

$$X * (Y + Z) = X$$

(5) Distributivity :

$$X + (Y * Z) = (X + Y) * (X + Z)$$

And $X * (Y + Z) = (X * Y) + (X * Z)$

(6) Complement :-

If $X \in S$, then there exists $\overline{X}$ in S such that

$$\overline{\overline{X}} = X$$

(7) Identity :

There exists e such that

$$X + e = e + X = X, \forall X \in S$$

(8) De Morgan Laws :-

$$(\overline{X + Y}) = \overline{X} * \overline{Y}$$

$$(\overline{X * Y}) = \overline{X} + \overline{Y}$$

This system forms a distributive lattice under operation + and $*$, But it is not a Boolean algebra because the laws of Boolean algebra.

$$X + \overline{X} = 1 \text{ and } X * \overline{X} = 0$$

are not true in fuzzy algebra.

Hence every Boolean algebra is a fuzzy algebra but converse is not necessarily true. Obviously, the class of fuzzy sets on any universe X forms a fuzzy algebra w.r. to. Union, intersection and complement as defined earlier. The identity element being the fuzzy null set ϕ and fuzzy whole set X.

<u>Fuzzy Uncertainty</u> :-

Suppose a research paper is sent to n independent referees for their opinions for the acceptability of the paper for publication in a particular journal. Each referee is asked to grade

the paper at some point in the scale 0, 0.1, 0.2, 0.3, ..., 0.9, 1.0. Where grade 0 means that the paper is completely unacceptable and there is no uncertainty in the mind of the referee about it and grade 1 means that the paper is completely acceptable and there is again no uncertainty in the mind of referee about it. Grade 0.1 means that the paper is almost unacceptable. But there are some good features in the paper which create some uncertainty in the mind of the referee. Similarly grade 0.9 means that the paper is almost acceptable, but there are some undesirable features which create some uncertainty in the mind of referee about its acceptability.

As far as the editor of the journal is concerned grade 0 and 1 give him a clear indication. But grades 0.1 and 0.9 create a certain degree of uncertainty in his mind which is the

same in both cases.

Similarly grades 0.2 and 0.8, grades 0.3 and 0.7, grade 0.4 and 0.6 represent the same degree of uncertainty for the editor. We call this type of uncertainty as fuzzy uncertainty as distinguished from probabilistic uncertainty. This fuzzy uncertainty is maximum when the referee gives the grade 0.5 because the editor is completely uncertain whether to accept the paper or reject it.

If the grade X is 0 or 1, the fuzzy uncertainty is 0 and if the grade X is 0.5, the fuzzy uncertainty is maximum. As X increases from 0 to 0.5, the fuzzy uncertainty increases from, 0 to a certain maximum value and as X increases further from 0.5 to 1.0, the fuzzy uncertainty decreases from this maximum value to Zero. Thus the fuzzy uncertainty of grade X is a

function of X with the following properties :

I. $f(x) = 0$ when $X = 0$ or 1.

II. $f(x)$ increases as X goes from 0 to 0.5

III. $f(x)$ decreases as X goes from 0.5 to 1.0

IV. $f(x) = f(1-x)$

It is desirable that $f(x)$ is a continuous and differentiable function but it is not necessary. Now if the n referees give

Giving rise to fuzzy uncertainty $f(x_1), f(x_2), \cdots f(x_n)$

independently grades $X_1, X_2, X_3, \ldots\ldots X_n$, then the total fuzzy uncertainty is

$$f(X_1) + f(X_2) + \ldots\ldots\ldots\ldots + f(X_n)$$

C.E. Shanon in 1948 gave a formula for measuring uncertainty

For probability distribution

$P = (P_1, P_2, P_3, \ldots\ldots\ldots\ldots P_n)$, the function that satisfies

all conditions of fuzzy uncertainty is -

$$- K \sum_{i=1}^{n} p_i \log p_i,$$ where K is a positive constant

and $\sum_{i=1}^{n} P_i = 1,$

He called it as measure of entropy,

(i) This entropy is maximum when

$$p_1 = p_2 = p_3 = \dots\dots\dots = p_n = 1/n$$

(ii) This entropy is zero if one of p_i's is 1 and others are zero.

The graph given here is

$$f(x) = -x \log x - (1-x) \log (1-x),\ 0 \leq X \leq 1.$$

This gives the measure of fuzzy entropy

$$F(x) = - \sum_{i=1}^{n} \mu_A(x_i) \log [\, \mu_A(x_i) \,]$$

$$- \sum_{i=1}^{n} \{1 - \mu_A(x_i)\} \log \{1 - \mu_A(x_i)\}$$

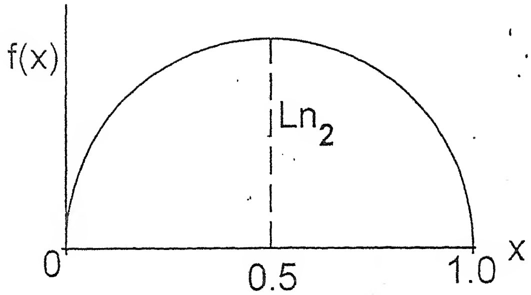


Fig. 1.5

And probabilistic entropy is

$$F(p) = -\sum_{i=1}^{n} p_i \log p_i - \sum_{i=1}^{n} (1-p_i) \log (1-p_i)$$

Some measures of fuzzy Entropy

(i) The simplest function satisfying the μ conditions of fuzzy function is

$$f_1(x) = x, \; 0 \le x \le ½$$

$$= 1 - x, \; ½ \le x \le 1$$

or $f_1(x) = 0.5 - | 0.5 - x |, \; 0 \le x \le 1$

Contd.....

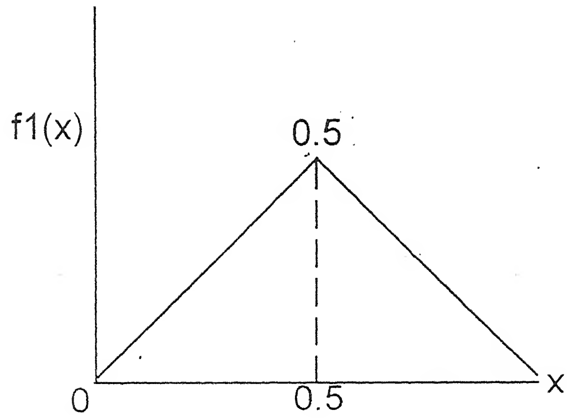


Fig. 1.6

This gives rise to the measure of fuzzy entropy

$$F_1 = \sum_{i=1}^{n} (0.5 - |0.5 - \mu_A(x_i)|)$$

The graph of $f_1(x)$, fig. 1.6 is continuous everywhere but it is not differentiable at $x = 0.5$

(ii) A function which is both continuous and differentiable and satisfies all the four conditions of fuzzy function is

$$f_2(x) = x - x^2, \; 0 \leq x \leq 1$$

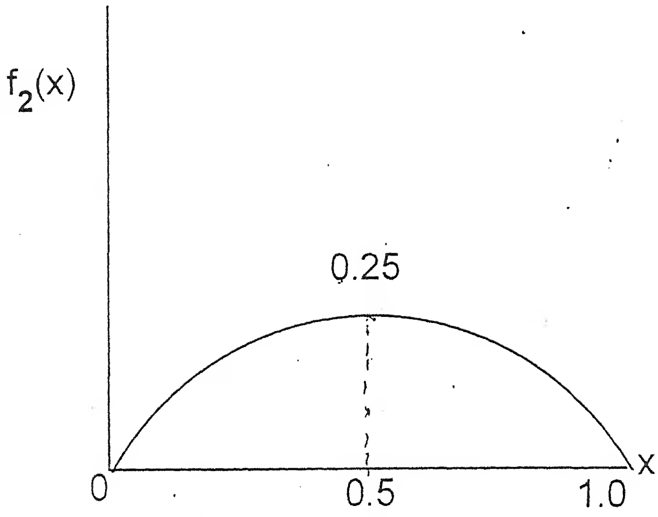


Fig. 1.7

This gives the measure of fuzzy entropy

$$F_2 = \sum_{i=1}^{n} \{ \mu_A(x_i) - \mu_A^2(x_i) \}$$

(iii) Consider the graph

$$f_3(x) = -x \ln x - (1-x) \ln (1-x),\ 0 \leq x \leq 1$$

$$= -x \log x - (1-x) \log (1-x),\ 0 \leq x \leq 1$$

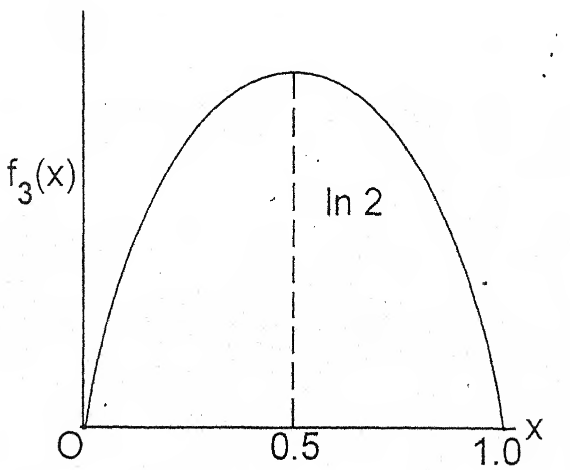


Fig. 1.8

This gives the measure of fuzzy entropy

$$F_3 = -\sum_{i=1}^{n} \{\mu_A(x_i) \ln \mu_A(x_i)\}$$

$$-\sum_{i=1}^{n} \{1-\mu_A(x_i)\} \ln \{1-\mu_A(x_i)\}$$

Measures of fuzzy Directed Divergence :-

Given the probability distributions $P = (P_1, P_2, ...P_n)$, $Q = (q_1, q_2, ..., q_n)$. We define the directed divergence of P from Q as a function D (P:Q) satisfying the following conditions.

(i) $D(P:Q) \geq 0$

(ii) $D(P:Q) = 0$ iff. $P = Q$

(iii) $D(P:U) = \text{Max } H(p) - H(p)$

Where $U = (1/n, 1/n, 1/n ....1/n)$ is the uniform distribution and H(P) is the measure of entropy.

Thus the measure of directed divergence corresponding to measure of Shanon entropy is given by :

$$\sum p_i \ln \frac{p_i}{q_i}$$

This measures how far the probability distribution P is from the probability distribution Q.

In the same way we define the directed divergence of fuzzy Set A from the fuzzy set B as a function D (A:B) which satisfies the conditions

(i) $D(A:B) \geq 0$

(ii) $D(A:B) = 0$ iff. $A = B$

(iii) $D(A:F) = \text{Max } H(A) - H(A) = H(F) - H(A)$

When F is the most fuzzy set i.e.,

$F = (½, ½, ½, \ldots\ldots\ldots\ldots ½)$

And H(A) is the fuzzy entropy of the set A.

One important measure of directed divergence of a fuzzy set A from the fuzzy set B is given by

$$= \sum_{i=1}^{n} \mu_A(x_i) \ln \frac{\mu_A(x_i)}{\mu_B(x_i)} +$$

$$\sum_{i=1}^{n} \{1-\mu_A(x_i)\} \ln \frac{\mu_A(x_i)}{\mu_B(x_i)}$$

Comparision of Probabilistic And fuzzy Measures of Entropy :-

1. $O \leq P_i \leq |$ for each i. Also $O \leq \mu_A(x_i) \leq |$ for each i.

2. $\sum_{i=1}^{n} P_i = 1$ for all probability distributions. But $\sum_{i=1}^{n} \mu_A(x_i)$ need not be Equal to unity and it need not even be the same for all fuzzy sets.

3. The probabilistic uncertainty measures how close the probability distribution $(P_1, P_2, \ldots\ldots\ldots P_n)$ is to the uni form distribution $(1/n, 1/n, \ldots\ldots\ldots 1/n)$. Fuzzy

uncertainty measures how close the fuzzy distribution is from the most fuzzy vector distribution (½, ½, ...½) and how for it is from the distribution of crisp sets.

4. $\mu_A(x_i)$ gives the same degree of fuzziness as $1-\mu_A(x_i)$ because both are equidistant from ½ and the crisp set values 0 and 1. However probabilities P and 1-P make different contributions to probabilistic uncertainty. As such while most measures of fuzzy entropy are of the form $\sum_{i=1}^{n} f\{\mu_A(x_i)\}+\sum_{i=1}^{n} f\{1-\mu_A(x_i)\}$ most measures of probabilistic entropy are of the form $\sum_{i=1}^{n} f(P_i)$. However same measures of probabilistic entropy can also be of the form $\sum_{i=1}^{n} f(P_i)+\sum_{i=1}^{n} f(1-P_i)$. in fact one such measure.

The fermi-Dirace measure was obtained as early as 1972 to get Fermi-Dirace distribution in statistical mechanics.

5. For each measure of probabilistic entropy or directed divergence, we have a corresponding measure of fuzzy entropy and fuzzy directed divergence and vice-versa.

6. The common properties arise from the consideration that both types of measures are based on measures of distance from (1/n, 1/n, ..., 1/n) in one case and from (1/2, 1/2, ...1/2) in the other.

7. The dissimilarity arises because while $\sum_{i=1}^{n} P_i = 1$, $\sum_{i=1}^{n} \mu_A(x_i)$ is not 1.

8. Principle of maximum entropy has been well established, while the principle of maximum fuzziness has yet to be studied.

Fuzzy identical sets,Fuzzy Equivalent sedts, standard fuzzy sets and measures of Directed divergence :-

Let us consider the following fuzzy sets.

A = (.3, .4, .8, .9), B = (.3, .4, .8, .9)

C = (.7, .6, .2, .1), D = (.7, .4, .8, .1)

E = (.3, .4, .2, .1).

The sets A and B are identical, since $\mu_A(x_i) = \mu_B(x_i)$

Again, .3 & 7; .6 and .4; .8 and .2; .9 and .1 give the same degree of fuzziness. As such sets A, B, C, D, E have the same fuzziness and same fuzzy entropy.

Two sets A and C are said to be fuzzy - equivalent if

$\mu_C(x_i)$ = either $\mu_A(x_i)$ or 1- $\mu_A(x_{.i})$. $\forall$ i

Thus sets A, B, C, D, E are fuzzy - Equivalent A & C

are fuzzy. Equivalent sets, yet the fuzzy directed divergence of A from C is not zero.

A Standard fuzzy set :- is that member of the family of fuzzy equivalent sets all of whose membership values are $\leq .5$.

Thus E is standard fuzzy set

Directed divergence of A from B is defined to be divergence between corresponding standard fuzzy sets $\bar{A}$ & $\bar{B}$.

Here $A = (.3, .4, .6, .8)$, $\quad \bar{A} = (.3, .4, .4, .2)$

$B = (.6, .4, .7, .9)$, $\quad \bar{B} = (.4, .4, .3, .1)$

$$\& \ I(A:B) \stackrel{def}{=} I(\bar{A}:\bar{B}) = .3 \log 3/4 + .4 \log 4/3 + .2 \log 2/1$$

$$+ .7 \log 7/6 + .6 \log 6/7 + .8 \log 8/9$$

$$= 0.088.$$

Fuzzy Numbers : A fuzzy nmumber is a convex fuzzy set A

on R (the set of real numbers) such that -

(i) There exists only one $x \in R$ satisfying

$$A(x) = 1.$$

(ii) A is piecewise continuous from R to [0,1].

Positive fuzzy number : A fuzzy number A is called positive if $A(X) = 0$ for all $X < 0$.

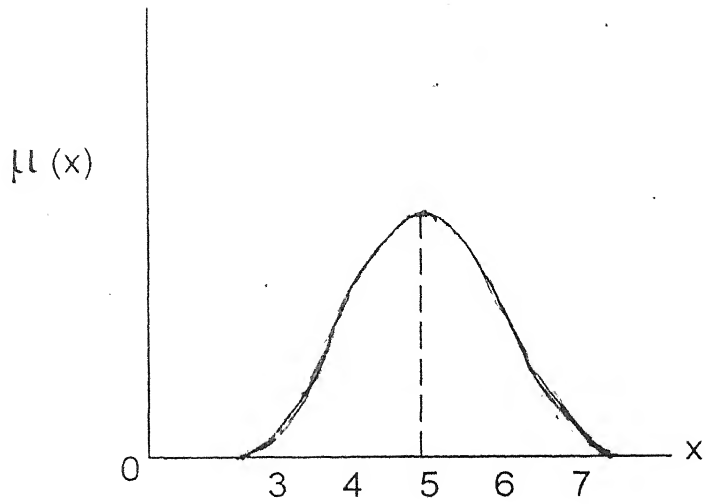
Negative fuzzy number : A fuzzy number A is called negative if $A(x) = 0$ for all $x > 0$.

Ex. The following fuzzy sets are fuzzy numbers :

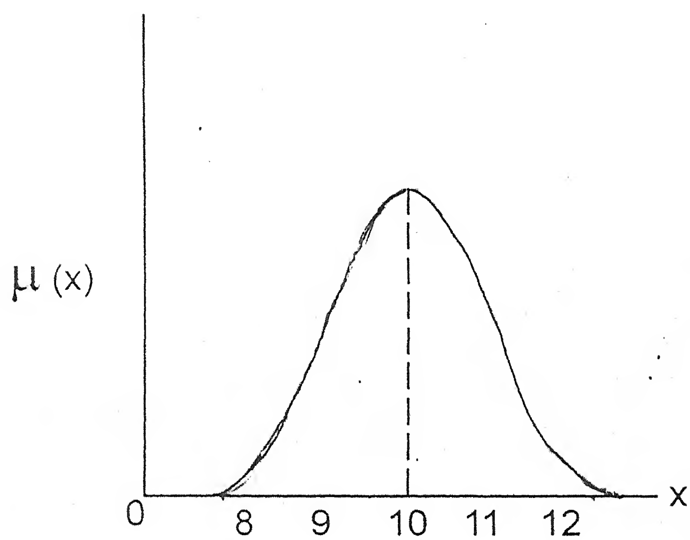
Approximately 5 = { (3, .2), (4, .6), (5,1), (6, .7), (7, .1)}

Approximately 10={ (8, .3), (9, .7), (10,1), (11,.7), (12,.3)}

But {(3,.8), (4,1), (5,1), (6,.7)} is not a fuzzy number because $\mu(4)$ and $\mu(5) = 1$.



Fuzzy number approx . 5.



Fuzzy number approx 10.

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CHAPTER - II

A Study of Fuzzy Relations

Abstract :

The concept of fuzzy set is generalisation of the concept of ordinary (crisp) set. Similarly the concept of fuzzy relations is generalisation of concept of relation, in set theory. Applications of fuzzy relations are wide spread and important. In present paper a study on fuzzy relations have been done for fuzzy binary relations. A generalization to n'.ary relations is straight forward.

Introduction :

Fuzzy relations are fuzzy sub sets of X x Y, i.e. mappings from $X \longrightarrow Y$.

They have been studied by a number of research workers, in particular by Zadeh [1965, 1971],

Kauffmann [1975] and Rosenfeld [1975].

We make a parallel study of different type of relations of ordinary set theory with reference to fuzzy set theory.

Def. 2.1 : A binary fuzzy relation :

A fuzzy set $\tilde{R} = \{(x,y), \mu_{\tilde{R}}(x,y) : (x, y) \in X \times Y\}$.

is called a binary fuzzy relation on X x Y.

Def. 2.2 : n' ary fuzzy relation :

An n' ary fuzzy relation is a fuzzy set on $X_1 \times X_2 \times X_3 \ldots\ldots \times X_n$.

Example 2.1.

Let X = Y = R (The set of real numbers), $\tilde{R}$: = "Considerably larger than". The membership function of the fuzzy relation, which is, of course, a fuzzy set on X x Y can then be

$$\mu_{\tilde{R}}(x,y) = \begin{cases} 0\ , \text{ for } x \le y \\ \dfrac{(x-y)}{10y} \text{ for } y < x \le 11\,y \\ 1\ , \text{ for } x > 11\,y \end{cases}$$

For discrete support's fuzzy relations can also be defined by matrices

Example 2.2.

Let $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3, y_4)$

$\tilde{R}$ = "x considerably larger than y".

	y_1	y_2	y_3	y_4
x_1	.8	1	.1	.7
x_2	0	.8	0	0
x_3	.9	1	.7	.8

and $\tilde{Z}$ = "y very close to x"

	y_1	y_2	y_3	y_4
x_1	.4	0	.9	.6
x_2	.9	.4	.5	.7
x_3	.3	0	.8	.5

Def. 2.3, : Let $X, Y \subseteq \mathbb{R}$ and

$\tilde{A} = \{ x, \mu_{\tilde{A}}(x) \} : x \in X$

and $\tilde{B} = \{ y, \mu_{\tilde{B}}(y) \} : y \in Y$ be two fuzzy sets.

Then $\tilde{R} = [\{(x,y), \mu_{\tilde{R}}(x,y)\} : (x,y) \in X \times Y]$ is a fuzzy relation on $\tilde{A}$ and $\tilde{B}$ if

$$\mu_{\tilde{R}}(x,y) \leq \mu_{\tilde{A}}(x), \quad \forall\ (x,y) \in X \times Y$$

$$\text{and } \mu_{\tilde{R}}(x,y) \leq \mu_{\tilde{B}}(y), \quad \forall\ (x,y) \in X \times Y$$

$$\text{i.e. } \mu_{\tilde{R}}(x,y) \leq \min. \{ \mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y) \}$$

Def. 2.4 : Let $\tilde{R}$ and $\tilde{Z}$ be two fuzzy relations in the same product space. Then union and intersection of $\tilde{R}$ and $\tilde{Z}$ is defined by

$$\mu_{\tilde{R} \cup \tilde{Z}}(x,y) = \max. \{ \mu_{\tilde{R}}(x,y), \mu_{\tilde{Z}}(x,y) \}, \quad (x,y) \in X \times Y$$

$$\mu_{\tilde{R} \cap \tilde{Z}}(x,y) = \min. \{ \mu_{\tilde{R}}(x,y), \mu_{\tilde{Z}}(x,y) \}, \quad (x,y) \in X \times Y$$

Example 2.3 :

Let $\tilde{R}$ and $\tilde{Z}$ be the two fuzzy relations defined in Example 2.2. The union of $\tilde{R}$ and $\tilde{Z}$, which can be interpreted as "x considerably larger or very close to y" is then given by

$$\tilde{R} \cup \tilde{Z}: \quad \begin{array}{c|cccc} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & .8 & 1 & .9 & .7 \\ x_2 & .9 & .8 & .5 & .7 \\ x_3 & .9 & 1 & .8 & .8 \end{array}$$

The intersection of $\tilde{R}$ and $\tilde{Z}$ is represented by

$$\tilde{R} \cap \tilde{Z}: \quad \begin{array}{c|cccc} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & .4 & 0 & .1 & .6 \\ x_2 & 0 & .4 & 0 & 0 \\ x_3 & .3 & 0 & .7 & .5 \end{array}$$

Def. 2.5 :

Let $\widetilde{R} = [\{(x,y), \mu_{\widetilde{R}}(x,y)\} : (x,y) \in X \times Y]$

be a fuzzy binary relation.

The first projection of R is then defined as

$$\widetilde{R}^{(1)} = \{(x, \max_{y} \mu_{\widetilde{R}}(x,y))\} : (x,y) \in X \times Y$$

The second projection is defined as

$$\widetilde{R}^{(2)} = \{(y, \max_{x} \mu_{\widetilde{R}}(x,y))\} : (x,y) \in X \times Y$$

And the total projection as

$$\widetilde{R}^{(T)} = \max_{x}.\ \max_{y}.\ \{(\mu_{\widetilde{R}}(x,y) : (x,y) \in X \times Y\}$$

Example 2.4. Let $\widetilde{R}$ be a fuzzy relation defined by the following relational matrix. The first, second and total projections are

then shown at the appropriate places below :

		y_1	y_2	y_3	y_4	y_5	y_6	Ist Projection [$\mu_{R(1)}(x)$]
	x_1	.1	.2	.4	.8	1	.8	1
$\tilde{R}$:	x_2	.2	.4	.8	1	.8	.6	1
	x_3	.4	.8	1	.8	.4	.2	1
IInd Projection [$\mu_{\tilde{R}(2)}(y)$]		.4	.8	1	1	1	.8	$\boxed{1}$

Total projection $\tilde{R}^{(T)}$.

Definition 2.6.

Inverse of binary fuzzy relation :

The inverse of a binary fuzzy relation $\tilde{R}$ on X x Y denoted by $\tilde{R}^{-1}$ is a fuzzy relation on Y x X defined by

$$\tilde{R}^{-1}((y, x)) = \tilde{R}((x, y))$$

Definition 2.7.

Reflexivity :

Let $\tilde{R}$ be a fuzzy relation in X x X, then $\tilde{R}$ is called reflexive

If $\mu_{\tilde{R}}(x, x) = 1, \forall \quad x \in X$

Example 2.5 :

Let $X = (x_1, x_2, x_3, x_4)$, $Y = (y_1, y_2, y_3, y_4)$

The following relation "y is close to x" is reflexive

	y_1	y_2	y_3	y_4
x_1	1	0	.2	.3
x_2	0	1	.1	1
x_3	.2	.7	1	4
x_4	0	1	.4	1

Definition 2.8 : Symmetry.

A fuzzy relation $\tilde{R}$ is called symmetric

if $\tilde{R}(x,y) = \tilde{R}(y,x)$

The relation $\tilde{R}(x,y)$:

	y_1	y_2	y_3	y_4
x_1	0	.1	0	.1
x_2	.1	1	.2	.3
x_3	0	.2	.8	.8
x_4	.1	.3	.8	1

Definition 2.9 :

Transitivity :

A fuzzy relation $\tilde{R}$ is called (max-min.) transitive

if $\tilde{R} \circ \tilde{R} \subseteq \tilde{R}$

Let the fuzzy relation $\tilde{R}$ be defined as

$\tilde{R}$:

	x_1	x_2	x_3	x_4
x_1	.2	1	.4	.4
x_2	0	.6	.3	0
x_3	0	1	.3	0
x_4	.1	1	1	.1

Then $\tilde{R} \circ \tilde{R}$ is

	x_1	x_2	x_3	x_4
x_1	.2	.6	.4	.2
x_2	0	.6	.3	0
x_3	0	.6	.3	0
x_4	.1	1	.3	.1

Here we see that $\mu_{\tilde{R} O \tilde{R}}(x, y) \leq \mu_{\tilde{R}}(x, y)$ holds

For all $x, y \in X$. Therefore relation $\tilde{R}$ is transitive.

Conclusions :

The combinations of the above property, give

inte resting result for max.- min. composition -

1. If $\tilde{R}$ is symmetric and transitive,then $\mu_{\tilde{R}}(x, y) \leq \mu_{\tilde{R}}(x, x)$ for all $x, y \in X$.

2. If $\tilde{R}$ is reflexive and transitive, then $\tilde{R} \circ \tilde{R} = \tilde{R}$.

3. If $\tilde{R}_1$ and $\tilde{R}_2$ are transitive and $\tilde{R}_1 \circ \tilde{R}_2 = \tilde{R}_2 \circ \tilde{R}_1$, then $\tilde{R}_1 \circ \tilde{R}_2$ is transitive.

Def. 2.10.

Fuzzy Partial Ordering relation :

A fuzzy relation P on X x X is called fuzzy partial ordering relation on X x X if it is reflexive, Perfectly anti symmetric and min. transitive on X x X.

Def. 2.11.

Tolerence Relation :

A fuzzy relation on X x X is called tolerence relation if it is reflexive and symmetric fuzzy relation on X x X.

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CHAPTER - III

A Study on Fuzzy Sub groups and Fuzzy Normal Sub groups

3.1 Abstract :

The concept of Fuzzy group is generalisation of concept of ordinary group. Similarly the concepts of a fuzzy sub group and Fuzzy Normal Sub group are generalisation of concept of ordinary sub group and normal sub group.

In present paper a study of fuzzy sub groups and fuzzy normal sub groups have been done and some new results have been derived.

3.2 Introduction :

A rosenfeld introduced the concept of Fuzzy group and fuzzy sub group in 1971 in a research paper published in

J.Math, Anal., Appl. 35 (1971) Pages 512-517, Dr. M.K.Singh and S.Sharma developed this work and presented a paper, "Some theorems on fuzzy groups" at 2nd Annual conference of Purvanchal Academy of Science, Jaunpur. In this chapter we analyse the results of A. Rosenfeld under changed definitions of fuzzy sub group. We also derive some new theorems and results. We define the concept of fuzzy normal sub group and derive some interesting results.

3.3 <u>A Rosenfeld's Definition of fuzzy sub group</u> :

A fuzzy set H on a group G is called fuzzy sub group of G if

(i) $H(xy) \geq \text{Min}\{H(x), H(y)\}, \forall\, x, y \in G.$

(ii) $H(x^{-1}) \geq H(x), \forall\, x \in G.$

Where H is fuzzy sub set of G and H(x) is its membership function.

Note that when H is taken to be ordinary sub set of G;

then $H(x) = 1$ & $H(y) = 1$ where x and $y \in H$.

Therefore from the above definition $H(xy) = 1$ and

then $x\,y \in H$.

i.e. $\forall x, y \in H \Rightarrow x\,y \in H$.

Also $H(x^{-1}) = 1$ when $H(x) = 1$ (i.e. when $x \in H$)

i.e. $\forall x \in H \Rightarrow x^{-1} \in H$

Thus the definition of fuzzy sub group has been taken

in such a manner that when H is taken to be ordinary (Crisp)

Sub set of G, then H turns out to be a sub group of G.

Now we define fuzzy sub group of a group as under :

3.4 <u>Fuzzy sub group of a group</u> :

A non empty fuzzy subset of a group G is called a

fuzzy sub group if

(i) $H(xy) \geq H(x).\,H(y), \quad \forall x, y \in G$

(ii) $H(x^{-1}) \geq H(x), \ \forall\, x \in G$

Note that when H is taken to be ordinary (crisp) sub set of G; then H (x) = 1, H (y) = 1 where x, y $\in$ H. Therefore from above definition.

$H(x, y) = 1$ and then $xy \in H$

Also $H(x^{-1}) = 1$ when $H(x) = 1$. Therefore $x^{-1} \in H$

Therefore $\forall x, y \in H \Rightarrow xy \in H$

and $\forall x \in H \Rightarrow x^{-1} \in H$

Therefore H is a sub group of G.

Thus our definition of fuzzy sub group is such that when H is taken to be ordinary (crisp) sub set of a group G, then H turns out to be a sub group of G.

3.5. **Theorem :**

If H is a fuzzy sub group of a group G, then

(a) $H(x^{-1}) = H(x), \forall x \in G$

(b) $H(e) \geq H^2(x)$, where e is identity in G.

(c) $H(e) = 1$ if $H(x) = 1$ for atleast one element $x \in G$.

Proof : (a) $H(x) = H\{(x^{-1})^{-1}\} \geq H(x^{-1})$

i.e. $H(x) \geq H(x^{-1})$ (1) By definition

and $H(x^{-1}) \geq H(x)$ (2) of sub group

From (1) & (2), $H(x^{-1}) = H(x)$.

(b) Since $e = x\,x^{-1}$.

Therefore $H(e) = H(x, x^{-1}) \geq H(x).\, H(x^{-1}) = H^2(x)$.

Therefore $H(e) \geq H^2(x) \quad \forall x \in G$.

(c) From (b) it follows that

$H(e) \geq H^2(x) \quad \forall x \in G$.

Therefore if $H(x) = 1$ for atleast one $x \in G$

Then $H(e) = 1$

3.6 **Theorem** :

If H is a sub group of a group G, then the characteristic function $H(x) = 1$ when $x \in H$. & $H(x) = 0$ when $x \notin H$.

is a fuzzy sub group of G and conversely.

Proof : Case I :- Let $x, y \in H$.

Then $H(x) = 1$, $H(y) = 1$

If H is a sub group of G,

Then $x, y \in H \Rightarrow x\,y \in H$.

Therefore $H(x\,y) = 1$

Therefore condition $H(xy) \geq H(x).\,H(y)$ holds,

Again if $x \in H$ and $y \notin H$.

Then $H(x) = 1$ and $H(y) = 0$

Therefore $H(x).\,H(y) = 0$.

Therefore $H(x\,y) \geq H(x).\,H(Y)$ holds

Again if $x \notin H$, $y \notin H$

Then $H(x) = 0$, $H(y) = 0$

Therefore $H(xy) \geq H(x).\ H(y)$ holds

Therefore condition $H(xy) \geq H(x).\ H(y)$ holds in all cases.

Case II : Suppose H(x) is a characteristic function of H in G and H is fuzzy sub group of G.

If $x, y \in H$ then $H(x) = 1 = H(y)$

And if $H(xy) \geq H(x), H(y) = 1$

Then $H(xy) = 1$

Therefore $xy \in H$

Again if $x \in H$, $H(x) = 1$

and since $H(x^{-1}) = H(x) = 1$

Therefore $x^{-1} \in H$

Therefore H is a sub group of G.

3.7 Theorem : The intersection of two fuzzy sub groups of a group is also a fuzzy sub group of that group.

Proof : Let H_1 and H_2 be two fuzzy sub groups of a group G.

Now Let x, y $\in$ G.

$$\bigcap_i H (xy) = H_1 (xy) \cap H_2(xy) = \text{Inf.} \{ H_1 (xy), H_2(xy) \}$$

$$\geq \text{Inf} \{ H_1(xy). H_2(xy) \}$$

$$\geq [\{\text{Inf.}\{H_1(x), H_2(x)\}] [\text{Inf.}\{H_1(y), H_2(y)\}$$

$$= \{ H_1(x) \cap H_2(x) \}. \{H_1(y) \cap H_2(y) \}$$

$$= \bigcap_i H_i(x). \bigcap_i H_i(y)$$

Again $H_i (x^{-1}) = \underset{i}{\text{Inf.}} H_i (x^{-1}) \geq \underset{i}{\text{Inf.}} H_i (x)$

$$= \cap H_i (x)$$

Therefore $H_1 \cap H_2$ is a fuzzy sub group of G.

Cor. Intersection of n fuzzy subgroup's H_1, H_2, ..., H_n of

a group G is also a fuzzy sub group of that group .

Def. 3.8 **Proper Fuzzy Sub group** :

The fuzzy sub group H of a group G is proper fuzzy sub group of G if $H(x) \neq 1$ for atleast one element $x \in G$.

i.e. if $H \neq G$.

Th. 3.9 : A Group can not be union of two proper fuzzy sub groups.

Proof : Let G be a group, If possible let A and B be two proper fuzzy subgroups of G such that $A \cup B = G$.

Then $A(x) = 1$ or $B(x) = 1, \forall x \in G$.

Let $x_1, x_2 \in G$ such that

$$A(x_1) = 1, B(x_1) < 1$$

$$\text{And } A(x_2) < 1, B(x_2) = 1$$

Now we prove that $A(x_1x_2) \neq 1$ and $B(x_1 x_2) \neq 1$

For otherwise, Let $A(x_1x_2) = 1$ then $A(x_2) = A(x_1^{-1}.x_1.x_2)$

$$= A\{(x_1^{-1}(x_1.x_2)\} \geq A(x_1^{-1}).\ A(x_1x_2)$$

$$= 1 \text{ as } A(x_1^{-1}) = 1 \text{ and } A(x_1.x_2) = 1.$$

Therefore $A(x_2) = 1$

Which is against our supposition.

Again if $B(x_1x_2) = 1$.

Then we get $B(x_1) = 1$ which is also against our supposition.

Therefore $AUB \neq G$.

Theorem 3.10 : If A is a fuzzy sub group of a group G then

$A(xy^{-1}) \geq A(x).\ A(y), \forall x, y \in G.$

And conversely if $A(e) = 1$ and $A(xy^{-1}) \geq A(x)\ A(y)$

$\forall x, y \in G$. Then A is a fuzzy sub group of G.

Proof : Case I. Let A be fuzzy sub group of a group G

Now $A(xy^{-1}) \geq A(x).A(y^{-1})$ (By def. of fuzzy sub group)

$= A(x)\ A(y)$ since $A(y^{-1}) = A(y)$ Proved

Case II. If $A(e) = 1$ and $A(xy^{-1}) \geq A(x)\,A(y)$

Then $A(x^{-1}) = A\,(ex^{-1})$

$\geq A(e)\,A\,(x^{-1}) = 1.A(x) = A(x)$

i.e. $A(x^{-1}) \geq A(x)$

And $A(xy) = A\,\{\,x\,(y^{-1})^{-1}\,\}$

$\geq A(x).\,A(y^{-1})$

$\geq A(x).\,A(y)$

Therefore A is a fuzzy sub group of G.

Def. 3.11 : **Fuzzy Normal Sub group**:

The Fuzzy sub group H of a group G is called fuzzy normal sub group if

$$H\,(xyx^{-1}) \geq H\,(y)\,\forall\, x,\, x^{-1} \in G.\ \forall\, y \in H.$$

Note : When H is ordinary sub group of a group G.

Then $y \in H \implies H\,(y) = 1$

Therefore $H(xyx^{-1}) \geq H(y)$

$$\geq 1$$

Therefore $H(xyx^{-1}) = 1$

$\Rightarrow$ $xyx^{-1} \in H$, which is definition of ordinary normal sub group

Th. 3.12 : Intersection of two fuzzy normal subgroups H_1 and H_2 of a group G i.e. $H_1 \cap H_2$ is also a fuzzy normal sub group of G.

Proof : If H_1 and H_2 are fuzzy normal sub groups of G then H_1 and H_2 are fuzzy sub groups of G. Therefore there intersection $H_1 \cap H_2$ is fuzzy sub groups of G. And since H_1 and H_2 are normal in G, therefore $H_1(xyx^{-1}) \geq H_1(y)$ and $H_2(xyx^{-1}) \geq H_2(y)$

Now $(H_1 \cap H_2)(xyx^{-1}) = \text{Min}\{H_1(xyx^{-1}), H_2(xyx^{-1})\}$

$$\geq \text{Min.}(H_1(y), H_2(y)\}$$

$$= (H_1 \cap H_2)(y)$$

Therefore $H_1 \cap H_2$ is fuzzy normal subgroup of G.

Cor. In general intersection of finite number of fuzzy normal subgroups $H_1\ H_2 \ldots\ldots\ldots H_n$ of a group G is also a fuzzy normal sub group of G.

Th. 3.13 : If G is abelian group then every fuzzy sub group of G is fuzzy normal sub group of G.

Proof : Let H be a fuzzy subgroup of an abelian group G.

Now $\forall x, x^{-1} \in G$ and $y \in H$

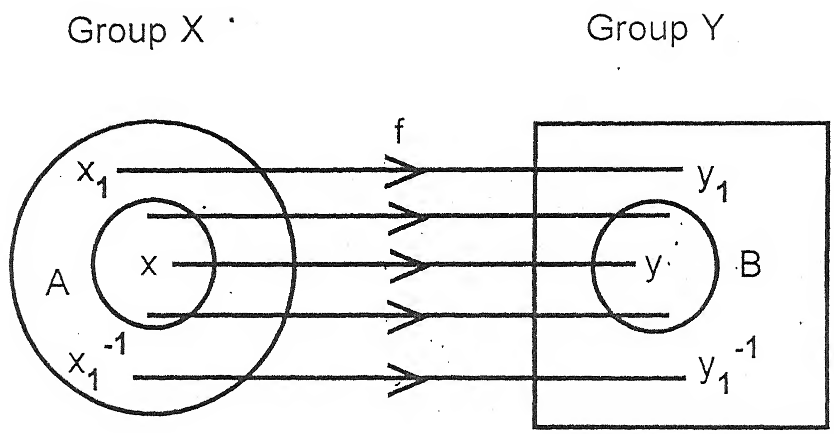
We have, $H(xyx^{-1}) = H(xx^{-1}y) = H(ey) = H(y)$

Therefore condition $H(xyx^{-1}) \geq H(y)$ is satisfied.

Hence H is a fuzzy normal subgroup of G.

Th. 3.14 : If f is an isomorphism from a group X onto Y and A is a normal subgroup of X, then image B of A under f is also a

fuzzy normal subgroup of Y.



Proof :

Let X and Y be two groups and

$f : X \longrightarrow Y$ is onto isomorphism (one one onto map)

If A is a fuzzy normal sub group of G, then we have to prove that its image under f i.e. B is also a fuzzy normal sub group of Y.

Let x_1, $x \in X$ and $f(x_1) = y_1$, $f(x) = y$

Now $B(y_1 \, y \, y_1^{-1}) = B[\, f(x_1).\, f(x).\, \{f(x_1)\}^{-1}\,]$

$= B[\, f(x_1).\, f(x).\, f(x_1^{-1})\,]$

$= B[f(x_1 . x x_1^{-1})]$ Since f is an isomorphism

$\geq B[f(x)]$

$= B(y)$

$\Longrightarrow B(y_1 y y_1^{-1}) \geq B(y) \quad \forall y_1, y_1^{-1} \in Y, \quad y \in B$

Therefore B is a fuzzy normal sub group of Y.

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CHAPTER - IV

A Study of fuzzy Vector Space

4.1 Abstract :

For a vector space addition of fuzzy sets and multiplication of a fuzzy set by a scalar have been defined. Under these operations, called internal and external compositions a fuzzy vector space is defined. Some interesting theorems and conclusions have been derived.

4.2 Introduction :

Since the inception of theory of fuzzy sets (1965, 1971), theoretical advances have been made in many directions of mathematics. A.K.Katsara and D.B.LIU (1977), introduced the concept of a new type of vector space which is a short of generalisation of ordinary vector space - called fuzzy

vector space. They tried to generalise ordinary vector space and defined some terms and drew conclusions.

In present chapter, we take X to be a vector space over a field K and examine the structure of fuzzy sets in X with respect to the operations of addition + in fuzzy sets and multiplication of fuzzy set by a scalar as defined below :

4.3 **Definition :** Addition of two fuzzy sets in a vector space :

Let X be a vector space over the field K, Let A & B be two fuzzy sets in X.

We define A+B to be a fuzzy set in X as,

$$(A+B)(y) = \sup_{\substack{x_1, x_2 \\ y = x_1 + x_2}} [\,\text{Min.}\,\{A(x_1), B(x_2)\}\,], \qquad y \in X$$

In general, if $A_1, A_2, \ldots\ldots\ldots, A_n$ are fuzzy sets in X

We define $A_1 + A_2 + \ldots\ldots + A_n$ to be a fuzzy set in X

as $(A_1 + A_2 + \ldots\ldots\ldots + A_n)(y) = \underset{x_1, x_2, \ldots, x_n}{\text{Sup}} [\text{Min.} \{ A_1(x_1), A_2(x_2),$

$\ldots\ldots\ldots, A_n(x_n)]$

$$y = x_1 + x_2 + \ldots\ldots + x_n, \quad y \in X$$

4.4 Definition : Multiplication of a fuzzy set by a scalar :

Let X be a vector space over a field $\bar{K}$. Let A be a fuzzy set in X and a be any scalar. We define aA as

$(aA)(y) = \underset{\substack{x \\ y = ax}}{\text{Sup}} A(x)$, if y = a x holds for some x in X & $y \in X$.

$= 0 \quad$ if $y \neq a x$ for any $x \in X$.

Ex. **4.1** For X = {1, 2, 3, 4, 5, ...10 }

& A = [(1, .1), (3, .5), (5, .6), (7, 1)]

B = [(1, .2), (3, 1), (8, .2)]

$$(A+B)(y) = \underset{\substack{x_1, x_2 \\ y = x_1 + x_2}}{\text{Sup.}} [\text{Min} \{ A(x_1), B(x_2) \}], \quad y \in X.$$

$(A+B)(2) = \underset{x_1, x_2}{\text{Sup.}} [\text{Min} \{ A(x_1), B(x_2) \}]$ Since 2=1+1

$$y = x_1 + x_2$$

$$= \text{Sup. } [\text{ Min } \{.1, .2\}] = .1$$

$$(A+B)(6) = \underset{x_1, x_2}{\text{Sup.}} [\text{ Min } \{ A(x_1), B(x_2) \}], \quad \text{Since } 6 = 1+5$$

$$= 2+4$$
$$= 3+3$$
$$= 4+2$$
$$= 5+1$$

$$y = x_1 + x_2$$

$$= \text{Sup } [\text{ Min } \{.1, 0\}, \text{ Min } \{0, 0\}, \text{ min } \{.5, 1\},$$

$$\text{Min } \{0, 0\}, \text{ Min } \{.6, .2\}]$$

$$= \text{Sup } [0, 0, .5, 0, .2] = .5$$

And for $a = 2$, $y = 2$

$$(a A)(y) = \underset{x}{\text{Sup}} A(x) \qquad \text{Since } 2 = 2.1$$

$$y = ax \qquad \therefore y = ax \text{ holds for } 2 \& 1 \in X.$$

$$= A(1) = .1$$

for $y = 5$, $a = 2$

$$(aA)(y) = \underset{x}{\text{Sup.}} A(x) \qquad \text{Since } 5 \neq 2x \text{ for any value of } x \in X$$

$$y = ax$$

$$= 0$$

for $y = 6$, $a = 2$

$(aA)(y) = \underset{X}{\text{Sup}}\, A(x)$ Since $6 = 2.3$ Where $3 \;\&\; 6 \in X$

$y = a.x$

$= A(3) = .5$ etc.

4.5 Definition :

Fuzzy vector space : A vector space X over a field K, equipped with additon + of two fuzzy sets and scalar multiplication • of a fuzzy set by a non zero scalar defined in X is called fuzzy vector space.

4.1 Theorem : If X be a vector space over the field K and A be a fuzzy set in X, then

(a) For any scalar $a \neq 0$

$(aA)(y) = A(a^{-1}y), \forall\, y \in X$

and for $a = 0$

$(aA)(y) = 0$ if $y \neq 0$

$= \underset{X}{\text{Sup}}\, A(x)$, if $y = 0$

(b) For all non zero scalar a, $(aA)(y) \geq A(y), \forall\, y \in X$

Proof : (a) Let $a \neq 0$

Now $(aA)(y) = \underset{x}{\text{Sup}}\, A(x) \quad y = a\,x$

$= A(a^{-1}y) \qquad \forall\, y \in X$

If $a = 0$ and $y \neq 0$ then $(aA)(y) = 0$ Since $y = 0.x$ does not hold.

If $a=0$ and $y=0$ then $(aA)(y) = \underset{x}{\text{Sup}}\, A(x),\ x \in X$ as $0 = 0.x$ holds.

for every $x \in X$.

(b) For all scalars $a \neq 0$. $(aA)(y) = \underset{\substack{x \\ y = ax}}{\text{Sup}}\, A(x), \quad x, y \in X$

Taking $y = ax$, we have $(aA)(ax) = \underset{\substack{x \\ ax=ax}}{\text{Sup}}\, A(x) = \underset{x}{\text{Sup}} A(x), \quad x \in X$

$\therefore (aA)(ax) \geq A(ax)$, for all $x \in X$

$\therefore (aA)(y) \geq A(y), \quad \forall\, y \in X$

4.2 Theorem : If X and Y are vector spaces over the same field K and f is a linear map from X to Y, then for Fuzzy sets A and B in X,

(a) $f(A+B) = f(A) + f(B)$

(b) $f(aA) = a f(A)$ for all scalars a.

Proof : Let $M = \{ f(x) : x \in X \}$

Let $a = \{ f(A+B) \}(y)$

and $b = \{ f(A) + f(B) \}(y)$

Case I : Let $y \notin M$

Then from extension principle

$\{ f(A+B) \}(y) = 0$, Since $f^{-1}(y) = \phi \therefore a = 0$

Let $y_1 + y_2 = y$, where $y_1, y_2 \in Y$

Then atleast one of y_1, y_2 is not in M for otherwise

$y_1 + y_2 \in M$ i.e. $y \in M$

$\therefore f(A)(y_1) = 0$, or $f(B)(y_2) = 0$

Now $b = \{ f(A) + f(B) \}(y) = \underset{y_1, y_2}{\operatorname{Sup}} [\operatorname{Min} \{ f(A)(y_1), f(B)(y_2) \}]$

$y = y_1 + y_2$

$= 0 \qquad \therefore a = b$

i.e. $f(A+B)(y) = \{ f(A) + f(B) \}(y)$ if $y \notin M$

Case II : If $y \in M$

$$a = \{ f(A+B) \}(y)$$

$$= \sup_{x} (A+B)(x)$$

$$y = f(x)$$

$\therefore$ for $\varepsilon > 0$, there exists $x \in X$

Such that $(A+B)(x) > a - \varepsilon$ and $f(x) = y$

i.e. $\text{Min}[A(x_1), B(x_2)] > a - \varepsilon$

For some x_1, x_2 with $x_1 + x_2 = x$ and

$f(x_1 + x_2) = f(x) = y$ i.e. $f(x_1) + f(x_2) = y$

and $b = \{ f(A) + f(B) \}(y)$

$$= \sup_{y_1, y_2} [\text{Min} \{ f(A)(y_1), \{ f(B) \}(y_2)]$$

$$y = y_1 + y_2$$

$\therefore b \geq \text{Min}[\{ f(A) \} \{ f(x_1) \}, \{ f(B) \} \{ f(x_2) \}]$

as $f(x_1) + f(x_2) = y$

Now $\{ f(A) \} \{ f(x_1) \} = \sup_{z} A(z)$

$$f(x_1) = f(z)$$

$\therefore \ \{f(A)\}\ \{f(x_1)\} \geq A(x_1)$

Similarly $\{f(B)\}\ \{f(x_2)\} \geq B(x_2)$

$\therefore \ b \geq \text{Min}\ \{A(x_1), B(x_2)\} \geq a - \varepsilon$, from above

but ε is arbitrary $\therefore \ b \geq a$ ... ... (1)

Again, for $\varepsilon > 0$, there exist $y_1, y_2 \in Y$ such that $y_1 + y_2 = y$

and Min $[\{f(A)(y_1), \{f(B)\}(y_2)] > b - \varepsilon$, we take $\varepsilon < b$

i.e. Min $[\sup_{x_1} A(x_1), \sup_{x_2} B(x_2)] > b - \varepsilon$

$y_1 = f(x_1)$. $\qquad y_2 = f(x_2)$

$\therefore$ There exists $x_1, x_2 \in X$, such that $y_1 = f(x_1)$

and $y_2 = f(x_2)$ and Min $\{A(x_1), B(x_2) > b - \varepsilon$.

$\therefore \ a > b - \varepsilon$

but ε is arbitrary $\therefore \ a \geq b$ ... ... (2)

$\therefore$ from (1) and (2) $a = b$

Proof of (ii) : Let $M = \{f(x) : x \in X$

Let $C = \{a\, f(A)\}\ (y)$

$d = \{f(aA)\}\ (y)$

Case I : If $y \notin M$ then $f^{-1}(y) = \phi$

Now $C = \{af(A)\}(y), \quad y \in Y$

$= \{f(A)\}(a^{-1}y)$ if $a \neq 0$

$= \sup\limits_{x} A(x)$

$a^{-1}y = f(x)$

but $f^{-1}(y) = \phi \quad \therefore \quad f^{-1}(a^{-1}y) = \phi \quad \therefore \quad C = 0$

and $d = \{f(aA)\}(y) = \sup\limits_{x} (aA)(x)$

$y = f(x)$

$= 0$ if $f^{-1}(y) = \phi \quad \therefore d = 0 \quad \therefore c = d$

Case (2) : Let $y \in M$ and $a \neq 0$

Then $C = \{af(A)\}(y) = \{f(A)\}(a^{-1}y)$ [Since $a \neq o$]

$= \sup\limits_{x} A(x)$

$a^{-1}y = f(x)$

$= \sup\limits_{x} (aA)(ax)$

$y = af(x) = f(ax) \quad = \sup\limits_{z} (aA)(z)$

$y = f(z) \; = \; d \qquad$ Therefore $c = d$.

If $a = 0$ and $y \neq 0$

$$C = \{ a\, f(A) \}(y) = 0$$

$$\text{and } d = \sup_{\substack{x \\ y = f(x)}} (aA)(x)$$

$= 0$ as when $y \neq 0$ then $x \neq 0$

If $a = 0$ and $y = 0$

$$C = \{ a\, f(A) \}(y) = \sup_{\substack{z \\ y = a z}} \{ f(A) \}(z),\ z \in Y$$

$$= \sup_{x} A(x),\ x \in X$$

$$d = \{ f(aA) \}(y) = \sup (aA)(x) \qquad (\text{Here } y = 0\ )$$

$$0 = f(x) \quad = \sup (aA)(o) \quad = \sup_{x \in X} A(x)$$

$$\therefore C = d$$

4·3 Theorem : If A and B are fuzzy set's in vector space X over a field K then for all scalar's a.

$$a(A + B) = aA + bB$$

Proof : Let f be a mapping from vector space X to X such that

$f(x) = ax, \forall\, x \in X$ and for any scalar a Let A and B be fuzzy set's in X.

By Extension Principle

$$\{f(A)\}(y) = \sup_{\substack{x \\ y = ax}} A(x), \text{ if } y = a\,x \text{ holds}$$

$$= 0 \qquad \text{if } y \neq ax \text{ for any } x$$

$$= (aA)(y) \qquad \therefore f(A) = aA$$

Similarly $f(B) = aB$

$\therefore\ aA + bB = f(A) + f(B) = f(A+B) = a(A+B)$

[Since A+B is fuzzy set in X and f is linear map]

4.6 Definition: If A is a fuzzy set in a vector space X and $x \in X$, we define $x + A$ as

$x + A = \{x\} + A$.

4.7 **Definition : Fuzzy sub space of vector space.**

The fuzzy set A in a vector space X is called fuzzy sub space of X if

(i) $A + A \subset A$ (ii) $aA \subset A$ for every scalar $a \neq 0$

Condition (ii) implies that $aA = A$ when $a \neq 0$

4.3 Theorem : If A and B are fuzzy subspaces of vector space X. Then A + B and aA are fuzzy subspaces where a is any scalar $\neq 0$.

Proof : Let A, B are fuzzy subspaces of vector space X

Then $A + A \subset A$ and $B + B \subset B$

Therefore $A(x_1 + x_2) \geq \min. \{A(x_1), A(x_2)\}$, $x_1, x_2 \in X$

and $B(x_3 + x_4) \geq \text{Min.} \{B(x_3), B(x_4)\}$, $x_3, x_4 \in X$

now $(A+B)(z) = \sup_{x_1, x_2} [\text{Min} \{A(x_1), B(x_3)\}]$

Therefore $(A+B)(x_1 + x_3) \geq \text{Min} \{A(x_1), B(x_3)\}$

Similarly $(A+B)(x_2 + x_4) \geq \text{Min} \{A(x_2), B(x_4)\}$

Now $(A+B)(x_1 + x_2 + x_3 + x_4) = (A+B)\{(x_1 + x_2) + (x_3 + x_4)\}$

$$\geq \text{Min} \{A(x_1 + x_2), B(x_3 + x_4)\}$$

$$\geq \text{Min} [\text{Min} \{A(x_1), A(x_2)\},$$

$$\text{Min} \{B(x_3), B(x_4)\}]$$

$$= \text{Min} \{A(x_1), A(x_2), B(x_3), B(x_4)\}$$

$\therefore \quad A + B + A + B \subset A + B \qquad \ldots \qquad \ldots \qquad (1)$

Again $K(A+B) = KA + KB$ where K is scalar $\neq o$

$= A + B \subset A + B$

$\therefore \quad K(A + B) \subset A + B \qquad \ldots \qquad \ldots \qquad (2)$

$\therefore$ from (1) and (2), $A + B$ is fuzzy subspace of X.

Proof of (ii) Part :

for $a \neq 0$, we have

$aA + aA = A + A \qquad A = aA \qquad \ldots \qquad \ldots \qquad (1)$

And for any scalar $K \neq 0$, $K(aA) = KA \qquad A = KA \ldots (2)$

$\therefore$ aA is fuzzy subspace.

4.4 Theorem : If A and B are fuzzy subspaces of vector space X then for non zero scalars a, b, $aA + bB$ is also fuzzy subspace.

4.5 Theorem : If A is a fuzzy set in a vector space X and for all non-zero scalars K, m

$KA + mA \subset A$

Then A is a subspace of X .

Proof : **Suppose $KA + mA \subset A$** **K, m are non-zero scalars**

Putting $K = m = 1$ in above, we have,

$A + A \subset A$ ... ... (1)

Again putting $m = 0$, we get $KA + 0A \subset A$

i.e. $(KA + 0A)(x) \leq A(x)$

i.e. $\sup_{x_1, x_2} [\operatorname{Min}\{(KA)(x_1), (0A)(x_2)\}] \leq A(x)$

$$x = x_1 + x_2$$

But $(0A)(x_2) = 0$ in all cases except when $x_2 = 0$ and then

$(0A)(0) = \sup_y A(y)$ and $x_1 = x$

$\therefore$ $(KA + 0A)(x) = \operatorname{Min}\{(KA)(x), \sup_y A(y)\} \leq A(x)$

i.e. $\operatorname{Min}\{(KA)(x), \sup_y A(y)\} \leq A(x)$

$\therefore$ $(KA)(x) \leq A(x)$

$\therefore$ $KA \subset A$ $\therefore$ A is subspace of X.

4.6 Theorem : If (A_i), $i \in I$ is a family of subspaces of a vector space X then $A = \bigcap_i A_i$ is also a fuzzy subspace.

Proof : Let K, m be scalars and x, y $\in X$

$$\text{Now } A(mx+ky) = \underset{i \in I}{\text{Inf.}} A_i (mx+Ky)$$

$$\geq \underset{i \in I}{\text{Inf.}} [\text{Min} \{ A_i(x), A_i(y) \}]$$

$$= \text{Min} \{ \text{Inf.} A_i(x), \underset{i \in I}{\text{Inf.}} A_i(y) \}$$

$$= \text{Min} \{ A(x), A(y) \}$$

Hence by Theorems 4.4 and 4.5, A is fuzzy subspace of X.

4.7 Theorem : If X and Y are vector spaces over the same field K and f is linear map from X to Y, then

(i) f(A) is fuzzy subspaces of Y is A is fuzzy subspace of X.

(ii) f^{-1} (B) is fuzzy subspace of X if B is fuzzy subspace of Y.

Proof of (i). Let K, m be scalar's = 0, A is fuzzy sub-space of X and f is linear map from X to Y .

$$\text{Now } K f(A) + mf(A) = f(KA) + f(mA)$$

$$= f(KA+mA)$$

but $KA+mA \subset A$ Since A is fuzzy subspace of X

$$\therefore K f(A) + m f(A) \subset f(A)$$

$\therefore$ f (A) is fuzzy subspace of Y.

Proof of (ii) Let B be fuzzy subspace of Y

$$f^{-1}(B)(Kx + my) = B\{f(Kx + my)\}$$

$$= B\{K f(x) + m f(y)\}$$

$$\geq \text{Min}[B\{f(x), B\{f(y)\}]$$

$$= \text{Min}[\{f^{-1}(B)\}(x), \{f^{-1}(B)\}(y)]$$

$\therefore$ f^{-1} (B) is fuzzy subspace of X.

4.8 **Definition : Convex fuzzy set** -

The fuzzy set A in a vector space X over the field of real numbers is said convex fuzzy set if $aA + (1-a)A \subset A$ for all $a \in [0,1]$.

4.8 Theorem : If A is a fuzzy set in a vector space X then A is convex if and only if

$$A(ax + (1-a)y\} \geq \text{Min}[A(x), A(y)]$$

for all $a \in [0,1]$ and $x, y \in X$

4.9 Theorem : If X and Y are vector spaces over the same field K and f is linear map from X to Y and A is convex fuzzy set in X then f (A) is also convex fuzzy set in Y.

Proof : Let $f : X \longrightarrow Y$, be linear transformation from vector space X to vector space Y over a field K and A be a convex fuzzy set in X.

Now $a\, f(A) + (1-a)\, f(A) = f\,[aA + (1-a)\, A\,]$

$\subset f(A)$ [Since $aA+(1-a)\, A \subset A$]

Therefore f(A) is a convex fuzzy set in Y.

4.10. Theorem : If f is a linear map from a vector space X to vector space Y over the same field K and B is convex fuzzy set in Y, then $f^{-1}(B)$ in also convex fuzzy set in X.

Proof : Let $f : X \longrightarrow Y$, be a linear map and $a \in [0, 1]$

Let $M = a\, f^{-1}(B) + (1-a)\, f^{-1}(B)$

$\therefore\ f(M) = a\, f f^{-1}(B) + (1-a)\, f f^{-1}(B)$

$= a\, B + (1 - a)\, B \subset B$

$\therefore \; M \subset f^{-1}(B)$

i.e. $a f^{-1}(B) (1-a) f^{-1} (B) \subset f^{-1}(B)$

$\therefore$ f (B) is a convex fuzzy set in X.

4.11 Theorem : If $[A_i]_{i \in I}$ is a family of convex fuzzy sets in vector space X. Then $\bigcap_{i \in I} A_i$ is also a convex fuzzy set in X.

Proof : Let $\{ A_i \}_{i \in I}$ be a family of convex fuzzy sets in vector space X over the field K.

$\therefore \; a A_i + (1 - a) A_i \subset A_i , \qquad a \in [0, 1], i \in I$

i.e. $A_i \{ ax (1-a) y \} \geq \text{Min} \{ A_i (x), A_i (y) \}$

Let $A = \bigcap_{i \in I} A_i$

Now $A (y) = \inf_i A_i (y) , \qquad \forall y \in X$

$A (a x + (1 - a)y \} = \inf_i A_i \{ax + (1-a)y \} \geq \text{Inf}_i \text{ Min} \{A_i (x),A_i (y)\}$

$= \text{Min} \{ \text{Inf}_i A_i (x), \text{Inf}_i A_i (y) \}$

$= \text{Min} \{ A (x), A (y) \}$

Therefore A is a convex fuzzy set in X.

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CHAPTER - V

A Study of fuzzy Topology

5.1 Abstract :

Fuzzy topology is defined. It is in fact a sort of generalisation of ordinary topology. A study has been made to examine how far the results of ordinary topology hold good in case of fuzzy topology.

5.2 Introduction :

C.L. Chang (1968) [3] defined a new type of topology which he called fuzzy topology. He defined fuzzy topology as under :

Definition : Let X be a non empty set and $\mathfrak{J}$ be a collection of fuzzy sets in X such that

$T_{1)}$ ϕ and $X \in \mathfrak{J}$

$T_2)$ If A and B $\in \mathfrak{J}$, then A $\cap$ B $\in \mathfrak{J}$.

$T_3)$ If $A_i \in \mathfrak{J}$ for each i $\in$ I, the index set, then

$$\bigcup_{i \in I} A_i \in \mathfrak{J}$$

i.e. $\mathfrak{J}$ is closed with respect to arbitrary union and finite intersection of fuzzy sets in $\mathfrak{J}$.

The pair $(X, \mathfrak{J})$ is called fuzzy topological space.

Note : When the elements of $\mathfrak{J}$ are ordinary sets, then $\mathfrak{J}$ is a topology on X.

The elements of $\mathfrak{J}$ are called fuzzy open sets. A fuzzy set in X is called closed fuzzy set if its complement is fuzzy open set.

For fuzzy topology we write fts in short.

Indiscrete fuzzy topology :-

A fuzzy topology $\mathfrak{J}$ having ϕ and X only is called

Indiscrete fuzzy topology.

Discrete fuzzy topology : A fuzzy topology having all fuzzy sets in X is called Discrete fuzzy topology.

5.3 **Definition : Comparision of topologies**

If $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are fuzzy topologies on X, we say that $\mathfrak{J}_1$ is coarser (or weaker) than $\mathfrak{J}_2$ or equivalently, $\mathfrak{J}_2$ is finer (or stronger) than $\mathfrak{J}_1$, If $\mathfrak{J}_1 \subset \mathfrak{J}_2$ and we write $\mathfrak{J}_1 < \mathfrak{J}_2$.

Like topology we have in fuzzy topology the following theorems :

5.1 Theorem : The intersection of two fuzzy topologies $\mathfrak{J}_1$ and $\mathfrak{J}_2$ on a set X is also a fuzzy topology on X.

5.2 Theorem : The intersection of finite number of fuzzy topologies $\mathfrak{J}_1$, $\mathfrak{J}_2$, ... $\mathfrak{J}_n$ is also a fuzzy topology.

5.3 Theorem : If P = { $\mathfrak{J}_i$ } be the family of all fuzzy

topologies on a non empty set X. then (P, $\leq$) is a complete lattice.

Proof : Since the least member is here indiscrete fuzzy topology on X and a greatest member is the discrete fuzzy topology on X.

5.4 Theorem : If (X, $\mathfrak{J}$) is fuzzy topological space then

(1) ϕ and X are fuzzy close set.

(2) Any arbitrary intersection of fuzzy close set in X is also a fuzzy close set in X.

(3) Union of any two fuzzy close sets in X is a fuzzy close set in X.

Note : Various concepts analogus to concepts defined in topology are so defined in fuzzy topology that when fuzzy set referred to are taken to be ordinary set's, the definitions fit in for the topology.

5.4 Definition : **Neighbourhood of a point.**

The fuzzy set N_x in fts (X, $\mathfrak{J}$) is called fuzzy neighbourhood of $x \in X$ if there exists a fuzzy open set $G \in \mathfrak{J}$, Such that $G \leq N_x$ and $G(x) = N_x(X) > 0$ obviously each open fuzzy set A is Nbhd of the point x at which A (X) > 0.

5.5 Definition : **Neighbourhood of a fuzzy set.**

The fuzzy set H in fts (X, $\mathfrak{J}$) is called a nbhd of fuzzy set A in X if there exists open fuzzy set $G \in \mathfrak{J}$.

Such that $A \leq G \leq H$

5.5 Theorem : The fuzzy set A in fts (X, $\mathfrak{J}$) is an open fuzzy set if and only if for every x satisfying $A(x) > 0$ there is N_x such that $N_x \leq A$ and $N_x(x) = A(x)$.

Proof : Suppose A is open fuzzy set in fts (X, $\mathfrak{J}$) and for $x \in X$, $A(x) > 0$. Then A(x) is also fuzzy Neighbourhood of x.

Taking $N_x = A$, we have

$$N_x \leq A \text{ and } N_x(x) = A_x \qquad \ldots \qquad \ldots \qquad (1)$$

Again suppose for each x where $A_x > 0$, $N_x \leq A$ Such that

$$N_x(x) = A_x$$

Let $G = \text{Sup.}\{\text{Open } N_x \leq A\}$

Then $G \in \mathfrak{J}$. and $G = A$

5.6 Definition : **<u>Interior of a fuzzy set.</u>**

The interior of a fuzzy set A in fts $(X, \mathfrak{J})$ denoted by A^0 is the least upper bound of all fuzzy set's B contained in A Such that A is nbhd of B.

i.e. $A^0 = \text{Sup } B$, $B \leq A$ and there is an open fuzzy set $G \in \mathfrak{J}$. Such that $B \leq G \leq A$. Such a set B is called interior fuzzy set of A.

5.7 Definition : **<u>Closure of a fuzzy set.</u>**

The closure of a fuzzy set A in fts $(X, \mathfrak{J})$ denoted as

$\bar{A}$ is the greatest lower bound of all fuzzy closed sets containing A i.e. $\bar{A} = \text{Inf.}\{B : B \geq A \text{ and } 1 - B \in \mathfrak{J}\}$.

The following theorems analogus to theorems on interior and closure of sets in topology hold.

5.6 Theorem : The interior $\mathring{A}$ of a fuzzy set A in fuzzy topology is the largest fuzzy open set contained in A and the fuzzy set A is open if and only if $A = \mathring{A}$.

Proof : Let $\mathring{A}$=Sup B where $B \leq A$ and there is an open set G

Such that $B \leq G \leq A$

Therefore Sup. $B \leq$ Sup. $G \leq A$

If Sup $G = G_1$ Therefore $\mathring{A} \leq G_1 \leq A$

But $G_1 \leq \mathring{A}$ for G_1 is interior fuzzy set.

Therefore $G_1 \leq$ least upper bound of interior fuzzy set of $A = \mathring{A}$

Therefore $\mathring{A} = G_1$ And Since G_1 = Sup G where G is open set

Therefore $\mathring{A}$ is the largest fuzzy open set contained in A.

Again If A is open then $A \leq \mathring{A}$ Also we have $\mathring{A} \leq A$

Therefore $A = \mathring{A}$

Conversely If $\mathring{A} = A$ then since $\mathring{A}$ is open, A is also open set.

5.7 Theorem : If $\bar{A}$ is the closure of a fuzzy set A in fuzzy topological space (X, $\mathfrak{J}$) then

$\bar{A}$ is closed set and is the least closed fuzzy set $\geq A$

Also A is closed if and only if $A = \bar{A}$

Proof : Let A be fuzzy set in fts (X, $\mathfrak{J}$). Then closure of A denoted by $\bar{A}$ is given by

$$\bar{A} = \inf. B, B > A \text{ and } 1 - B \in \mathfrak{J}$$

$\bar{A}$ is closed fuzzy set in X and $\bar{A} \geq A$

Since $\bar{A}$ is infimum of $B \geq A$ such that $1\text{-}B \in \mathfrak{J}$

Therefore $\bar{A}$ is the least closed fuzzy set $\geq A$

If $A = \bar{A}$ then A is closed fuzzy set since $\bar{A}$ is closed fuzzy set.

Again if A is closed fuzzy set, then

$\bar{A}$ = Inf. closed fuzzy set B, $B \geq A$

Now since A is closed and $A \geq \bar{A}$ Therefore $\bar{A} \leq A$

Also we have $\bar{A} \geq A$ Therefore $A = \bar{A}$.

5.8 Theorem : In a fuzzy topological space $(X, \mathfrak{J})$ the following hold :

(i) $\bar{\phi} = \phi$ (ii) $\bar{X} = X$

(iii) For each fuzzy set A in X, $A \leq \bar{A}$

(iv) For each fuzzy set A in X, $\bar{\bar{A}} = \bar{A}$

(v) For each pair of fuzzy sets A, B in X, $\overline{A \cup B} = \bar{A} \cup \bar{B}$

(vi) $\overline{A \cap B} \leq \bar{A} \cap \bar{B}$

(vii) If $A \leq B$, then $\bar{A} \leq \bar{B}$, $A, B \in \mathfrak{J}$

Proof : (i), (ii), (iii) follow from definition of closure.

(iv) Since $\bar{A}$ is closed fuzzy set, $\bar{\bar{A}} = \bar{A}$

(v) Since $\bar{A}$, $\bar{B}$ are closed fuzzy set therefore $\bar{A} \cup \bar{B}$ is closed fuzzy set. Also $\bar{A} \cup \bar{B} \geq A \cup B$. Therefore from the

definition of closure $\bar{A} \cup \bar{B} \geq \overline{A \cup B}$ ... ... (1)

Again $A \cup B \geq A$ Therefore $\overline{A \cup B} \geq \bar{A}$

And $A \cup B \geq B$ Therefore $\overline{A \cup B} \geq \bar{B}$

Therefore $\overline{A \cup B} \geq \bar{A} \cup \bar{B}$ ... ... (2)

From (1) and (2) $\overline{A \cup B} = \bar{A} \cup \bar{B}$

(vi) $\bar{A}$, $\bar{B}$ are closed fuzzy sets. $\bar{A} \geq A$, $\bar{B} \geq B$.

Therefore $\bar{A} \cap \bar{B}$ is a closed fuzzy set.

Also $\bar{A} \cap \bar{B} \geq A \cap B$

Therefore from the definition of closure

$\overline{A \cap B} \leq \bar{A} \cap \bar{B}$

(vii) $\bar{A}$ = Inf { closed fuzzy set C such that $C \geq A$ } .... (1)

and $\bar{B}$ = Inf { closed fuzzy set D such that $D \geq B$ } ... (2)

But $B \geq A$

Therefore every closed fuzzy set D satisfying (2), satisfies (1)

$\bar{A} \leq \bar{B}$

5.9 Theorem : In a fuzzy topological space $(X, \mathcal{J})$ the following hold :

(i) $\phi^0 = \phi$

(ii) $X^0 = X$

(iii) $\mathring{A} < A$.

(iv) for each fuzzy set A in X , $(\mathring{A})^0 = \mathring{A}$

(v) for each pair of fuzzy set A,B $\mathring{A} \cap \overset{0}{\mathring{B}} = (A \cap B)^0$

(vi) $\mathring{A} \cup \overset{0}{\mathring{B}} \leq (A \cup B)^0$

(vii) If $A \leq B$ then $\mathring{A} \leq \overset{0}{\mathring{B}}$.

Proof : (i), (ii), (iii),(iv) are obvious from definition.

(v) For fuzzy set A, B $\mathring{A}$, $\overset{0}{\mathring{B}}$ are open sets

Therefore $\mathring{A} \cap \overset{0}{\mathring{B}}$ is an open sets. Therefore $\mathring{A} \cap \overset{0}{\mathring{B}}$ is an open set. Also $\mathring{A} \leq A$, $\overset{0}{\mathring{B}} \leq B$.

Therefore $\mathring{A} \cap \overset{0}{\mathring{B}} \leq A \cap B$ ... .(1)

Again $A \cap B \leq A$

Therefore $(A \cap B)^0 \leq \mathring{A}$

Similarly $(A \cap B)^0 \leq \overset{0}{B}$

Therefore $(A \cap B)^0 \leq \mathring{A} \cap \overset{0}{B}$ ... ... (2)

Therefore from (1) and (2) $(A \cap B)^0 = \mathring{A} \cap \overset{0}{B}$

(vi) $\mathring{A} \leq A$, $\overset{0}{B} \leq B$. Therefore $\mathring{A} \cup \overset{0}{B} \leq A \cup B$

Also $\mathring{A} \cup \overset{0}{B}$ is an open set

Therefore $\mathring{A} \cup \overset{0}{B} \leq (A \cup B)^0$

The following relations between two operations of interior and closure of fuzzy set hold.

5.10 Theorem : For any fuzzy set A in fts $(X, \mathfrak{J})$

(i) $(1-A)^0 = 1 - \bar{A}$

(ii) $(\overline{1-A}) = 1 - \mathring{A}$

Proof : $1 - \bar{A} = 1 - \text{Inf}\{ D : 1 - D \in \mathfrak{J} \text{ and } D \geq A \}$

$= \text{Sup}\{ 1-D : 1-D \in \mathfrak{J} \text{ and } D \geq A \}$

$= \text{Sup}\{ C : C \in \mathfrak{J} \text{ and } C \leq 1-A \}$

Where $C = 1 - D$

$= (1 - A)^0$

Similarly $1-\mathring{A} = 1- \text{Sup}\{ B : B \leq G \leq A \text{ for some } G \in \mathfrak{J} \}$

$= \text{Inf.}\{ 1-B : B \leq G \leq A \text{ for some } G \in \mathfrak{J}\}$

$= \text{Inf.}\{ 1-B : 1-B \geq 1-G \geq 1-A, G \in \mathfrak{J}\}$

$= (\overline{1 - A})$

5.8 Definition : Fuzzy continuity -

A function f from fts $(X, \mathfrak{J})$ to the (Y, U) is said to be fuzzy continuous (F-continuous) If inverse image of U- open set is $\mathfrak{J}$ - open fuzzy set.

5.11 Theorem : If $(X, \mathfrak{J})$ and (Y, U) are fuzzy topological spaces, then the following statements are equivalent :

(1) The function $f : X \longrightarrow Y$ is f- continuous.

(2) The inverse image of every closed fuzzy set under f is closed fuzzy set.

(3) For every $x \in X$ and for every fuzzy nbhd N of f(x), $f^{-1}(N)$ is fuzzy nbhd of x.

(4) For every $x \in X$ and for every fuzzy nbhd N of f(x),there is fuzzy Nbhd M of x such that $f(M) \leq N$ and $M(x) = \{f^{-1}(N)\}(x)$.

(5) For every fuzzy set A in X, $f(\bar{A}) \leq \overline{\{f(A)\}}$.

(6) For every fuzzy set B in Y, $f^{-1}(B) \leq f^{-1}(\bar{B})$

In spite of all these credits in fuzzy topology, the following does not hold.

5.12 Theorem : A constant fuction from fts $(X, \mathfrak{J})$ to fts (Y,U) is not necessarily fuzzy continuous.

Compact fuzzy topological space

5.9 **Definition:Cover of fuzzy set:** A family F of fuzzy sets in X is said to be cover of fuzzy set B if. $B \subset \cup \{ A : A \in F \}$

5.10 **Definition:Open cover of fuzzy set:** A family F of open fuzzy sets of fts $(X, \mathfrak{I})$ is called open cover of fuzzy set B in X if $B = \cup \{A: A \in F$ and A is open fuzzy set $\}$

A sub family of F which covers a fuzzy set B is called sub cover of F.

5.11 **Definition : Compact fuzzy topological space :** C.L.Chang defines a fuzzy topological space $(X\ \mathfrak{I})$ to be compact if each open cover of X has a finite sub cover.

5.12 **Definition : Finite Intersection Property :** A family F of fuzzy sets is said to have finite intersection property if the intersection of the members of each finite sub family is non empty.

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CHAPTER - VI

Fuzzy integration and fuzzy differentiation

6.1 Abstract :

On the lines of Lebesgue measures and Lebesgue integrals, fuzzy measure is considered and a definition of fuzzy integral which is a generalization of Lebesgue integral is given. Also in analogy to fuzzy integration, fuzzy differentiation is defined. Differentiation of fuzzy functions is considered. We have considered fuzzy differentiation of a differentiable function.

6.2 Fuzzy Integration :

Introduction :

One of the first concepts of a fuzzy integral was put forward by Sugeno [1972, 1977]; who considered fuzzy measures and suggested a definition of a fuzzy integral which is a

generalization of Lebesgue integral : "From the view point of functionals, fuzzy integrals are merely a kind of non linear functionals while Lebesgue integrals are linear one's" [Sugeno 1977, p.92].

We shall focus our attention on approaches along the lines of Riemann Integrals. The main references for the following are Dubois and Prade [1980a, 1982b], Aumann [1965] and Nguyen [1978].

The classical concept of integration of a real valued function over a closed interval is generalized in two ways.

(i) The function can be fuzzy function which is to be integrated over a crisp interval, or

(ii) Function is crisp and interval is fuzzy.

6.3 <u>Integration of a fuzzy function over a crisp interval :</u>

Let the fuzzy function be L.R. type,

We shall therefore assume that, $\tilde{f}(x) = [f(x), s(x), t(x)]_{LR}$

This is a fuzzy number in LR representation for all $x \varepsilon [a, b]$; f, s, t are assumed to be positive integrable functions on [a, b].

Dubois and Prade [1980a, p. 109] have shown that under these conditions

$$\tilde{f}(a, b) = \left(\int_a^b f(x)dx, \int_a^b s(x)dx, \int_a^b t(x)\, dx \right)_{LR}$$

It is then sufficient to integrate the mean value and spread functions of $\tilde{f}(x)$ over [a,b] and the result will be again LR fuzzy number.

6.1 Ex. Let fuzzy function be $\tilde{f}(x) = [(f(x), s(x) t(x)]_{LR}$

with the mean function.

$f(x) = x^2$, the spread functions $s(x) = x/4$ and $t(x) = x/2$

$L(x) = 1/1+x^2$, $R(x) = 1/1+2|x|$

To determine the integral from a = 0 to b = 4; that is to compute $\int_1^4 f$.

According to above formula we compute

$$\int_a^b f(x)\,dx = \int_1^4 x^2 dx = 21$$

$$\int_a^b s(x)\,dx = \int_1^4 x/4\,dx = 1.875$$

$$\int_a^b t(x)\,dx = \int_1^4 x/2\,dx = 3.75$$

This yields the fuzzy number

$\tilde{f}(a, b) = (21, 1.875, 3.75)_{LR}$ as the value of the fuzzy integral.

One interesting property of fuzzy integrals is as under :-

6.4Th. **Let $\tilde{f}$ and $\tilde{g}$ be fuzzy functions whose supports are bounded.**

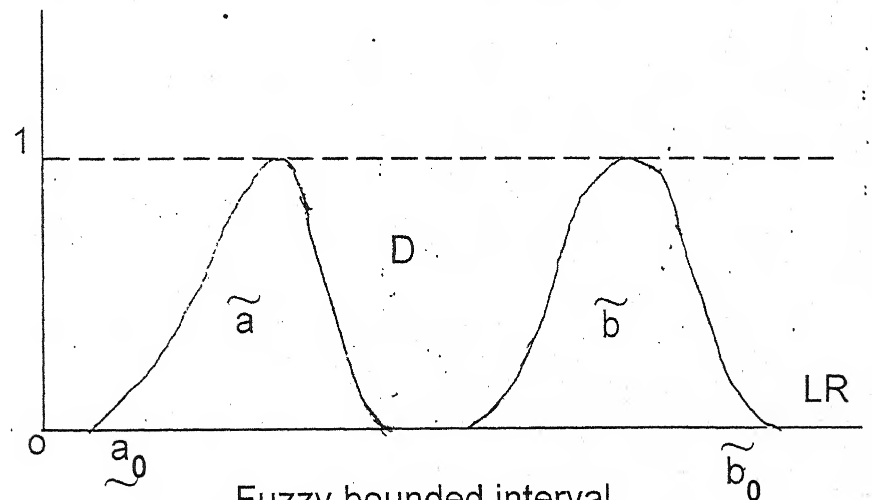
Then

$$\int_t (\tilde{f} \oplus \tilde{g}) \supseteq \int_t \tilde{f} \oplus \int_t \tilde{g} \quad \ldots \quad (1)$$

6.4 Integration of a (crisp) Real valued function over a fuzzy interval :-

Now, we consider a case for which Dubois and Prade [1982a, p.106] proposed a quite interesting solution :

A fuzzy domain of the real line R is assumed to be bounded by two normalized convex fuzzy sets the membership functions of which are $\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{b}}(x)$, respectively.



Fuzzy bounded interval

$\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{b}}(x)$ can be interpretted as degrees of confidence to which x can be taken to be a lower or upper bound of interval.

6.5 Definition :-

Let f be a real- valued function which is integrable in the interval $[a_0, b_0]$, then according to the extension principle the membership function of the integral $j = \int f$ is given by

$$\mu_{\int j} f(z) = \sup_{x,\, y\, \varepsilon\, j} \{ \text{Min}\ \mu_{\bar{a}}(x), \mu_{\bar{b}}(y) \}$$

$$Z \doteq \int_x^y f$$

6.2 Ex. Let $\tilde{a} = \{ (4, .8), (5, 1), (6, .4) \}$

$\tilde{b} = \{ (6, .7), (7, 1), (8, .2) \}$

$f(x) = 2, \qquad x\ \varepsilon\ [a_0, b_0] \quad = \quad [4, 8]$

Then $\int f(x)\, d\, x = \int_{\bar{a}}^{\bar{b}} 2\, d\, x = 2\, x \int_{\bar{a}}^{\bar{b}}$

The detailed computational results are :

[a, b]	$\int_a^b 2\,dx$	min. { $\mu_X(a), \mu_X(b)$ }
[4, 6]	4	.7
[4, 7]	6	.8
[4, 8]	8	.2
[5, 6]	2	.7
[5, 7]	4	1.0
[5, 8]	6	.2
[6, 6]	0	.4
[6, 7]	2	.4
[6, 8]	4	.2

Hence choosing maximum of the membership values for each value of the integral, yields.

$$\int f = \{ (0,.4), (2, .7), (4,1), (6,.8), (8,.2) \}$$

6.1 Proposition

Let f and g be two functions f, g : $I \rightarrow R$, integrable on I

[a, b]	$\int_a^b g(x)dx$	min.{μ_x (a), μ_x(b) }
[1, 3]	2	.7
[1, 4]	0	.8
[1, 5]	- 4	.3
[2, 3]	0	.7
[2, 4]	- 2	1.0
[2, 5]	- 6	.3
[3, 3]	0	.4
[3, 4]	- 2	.4
[3, 5]	- 6	.3

Hence choosing max. value of membership function for each value of the integral.

$\int f$ = { (0, .4), (2, .7), (4, .4), (6, 1), (10, .3), (12, .3) }

$\int g$ = { (-6,.3), (-4,.3), (-2, 1), (0,.8), (2, .7) }

$[a, b]$	$\int_a^b (f+g)$	$\min.\{\mu_x(a), \mu_x(b)\}$
[1, 3]	4	.7
[1, 4]	6	.8
[1, 5]	8	.3
[2, 3]	2	.7
[2, 4]	4	1.0
[2, 5]	6	.3
[3, 3]	0	.4
[3, 4]	2	.4
[3, 5]	4	.3

Hence choosing max. value of membership function for each value of the integral.

$$\int(f+g) = \{ (0, .4), (2, .7), (4, 1), (6, .8), (8, .3) \}$$

Applying the formula for the extended addition according to the extension principle.

$$\int_{\bar{a}}^{\bar{b}} f \oplus \int_{\bar{a}}^{\bar{b}} g = \{ (-6,.3),(-4,.3),(-2,.3),(0,.4),(2,.7)\ (4,1),(6,.8), (8,.3),(10,.3),(12,.3),(14,.3) \}$$

Therefore, we easily verify that

$$\int_{\bar{a}}^{\bar{b}} f \oplus \int_{\bar{a}}^{\bar{b}} g \supseteq \int_{\bar{a}}^{\bar{b}} (f+g)$$

6.6 <u>Fuzzy Differentiation</u> :

<u>Introduction</u> :

In analogy to fuzzy integration, Fuzzy differentiation is defined.

The results will depend on the type of function considered.

Differentiation of fuzzy functions is considered by Dubois and Prade [1980a, p. 116 and 1982b, p. 227]

Here we consider only fuzzy differentiation of a differentiable function.

$f : R \supseteq [a, b] \rightarrow R$. at a "fuzzy point".

"A fuzzy point" $\widetilde{X}_0$ [Dubois and Prade 1982b, p. 225] is a convex fuzzy subset of the real line R.

In present case, fuzzy point is considered for which the support is contained in the interval [a, b] , that is $S(\tilde{x}) \subseteq [a, b]$.

Such a fuzzy point can be interpreted as the possitibility distribution of a point x whose precise location is only approximately known.

The uncertainty of the knowledge about the precise location of the point induces an uncertainty about the derivative $f'(x)$ of a function $f(x)$ at this point. The derivative might be the same for several x belonging to [a, b].

The possibility of $f'(\tilde{X}_0)$ is therefore defined [Zadeh 1978] to be the supremum of the values of the possibilities of $f'(x) = t$, $x \in [a, b]$.

The "Derivative" of a real valued function at a fuzzy point can be interpreted as the fuzzy set $f'(\tilde{x}_0)$, the member-

ship function of which expresses the degree to which a specific

$f'(x)$ is, the first derivative of a function f at point $\tilde{X}_0$.

6.7 Definition :

The membership function of the fuzzy set "derivative of a real valued function at a fuzzy point $\tilde{X}_0$" is defined by the extension principle as

$$\mu_{f'(\tilde{x}_0)}(y) = \sup_{x \in f^{-1}(y)} \mu_{\tilde{x}_0}(x).$$

Where $\tilde{X}_0$ is the fuzzy number that characterizes the fuzzy location.

6.3 Ex. Let $f(x) = x^3$

and $\tilde{X}_0 = \{ (-1, .4), (0, 1), (1, .6) \}$

be a fuzzy location.

Since $f'(x) = 3x^2$

We obtain

$f'(\tilde{x}_0) = \{ (0, 1), (3, .6) \}$

as derivative of a real valued function at the fuzzy point $\widetilde{x}_0$.

For fuzzy differentiation, following propositions hold :

Proposition (1) : For the extended sum $\oplus$ of the derivative of two real valued functions f and g

$$f'(\widetilde{x}_0) \oplus g'(\widetilde{x}_0) \supseteq (f'+g')\widetilde{x}_0.$$

Proposition (2) : If f and g are continuous and both non decreasing or non increasing then -

$$f'(\widetilde{x}_0) \oplus g'(\widetilde{x}_0) = (f'+g')\widetilde{x}_0.$$

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CHAPTER - VII

A comparative Study of Fuzzy and Probabilistic Measures of Inromation

7.1 Abstract :

Different type of measures such as measures of entropy, measures of directed divergence and measures of symmetric divergence have been defined and discussed for probability distributions in previous years. Recently these measures have also been defined for fuzzy sets in similar manner. But the two sets of measures have different interpretations. A comparative study of these two sets of measures has been done in this paper.

7.2 Introduction :

Probabilistic measures of information for the probability distribution of the type

$$P = (P_1, P_2, P_3 \ldots\ldots\ldots P_n), \quad \ldots \quad \ldots \quad (1)$$

is defined where

$$0 \leq p_i \leq 1 \qquad i = 1, 2, \ldots\ldots n$$

$$\text{and} \quad \sum_{i=1}^{n} p_i = 1 \quad \ldots \quad \ldots \quad (2)$$

The probabilities are thus dependent and one can not be changed without changing one or more of the others.

Where as fuzzy measures of information are defined for fuzzy sets with fuzzy vectors of the type

$$\{\mu_A(x_1), \mu_A(x_2), \mu_A(x_3), \ldots\ldots \mu_A(x_n)\} \quad \ldots \quad (3)$$

Where $\mu_A(x_i)$ denotes the degree of membership of the element x_i of the set A.

If $\mu_A(x) = 0, \quad x \not\subset A.$

If $\mu_A(x) = 1, \quad$ x definitely belongs to A.

and $0 \leq \mu_A(x_i) \leq 1$, for i = 1,2, 3, ...n ... (4)

The sum $\Sigma\mu_A(x_i)$ is not the same for all fuzzy sets.

The values of $\mu_A(x_i)$ are inde[pendent and any one of them can be changed without affecting the others.

While mathematical forms of two sets of measures seem to be similar, yet they are quite different.

7.3 Measures of Entropy :

Among many measures defined for probability entropy and for fuzzy entropy, we take here the simplest measures.

$$E(P) = -\sum_{i=1}^{n} p_i \log p_i - \sum_{i=1}^{n}(1-p_i) \log (1-p_i) \quad \ldots \quad (a)$$

$$\text{and } E(A) = -\sum_{i=1}^{n} \mu_A(x_i) \log \{\mu_A(x_i)\}$$

$$-\sum_{i=1}^{n}\{1-\mu_A(x_i)\}\log\{1-\mu_A(x_i)\} \quad \ldots \quad (b)$$

respectively for the purpose of comparision.

(1) E(P)is maximum when $P_1 = P_2 = P_3 = ...=P_n = 1/n$.

E(A) is maximum when

$$\mu_A(x_1) = \mu_A(x_2) = \;........ = \mu_A(x_n) = ½$$

(2) E(P) is minimum when one of the p_i's is unity and others are zero just as for

$D_1 = (1, 0, 0 ....0)$,

$D_2 = (0, 1, 0, 0 ....0)$,

....

$D_n = (0, 0 ....0,1)$. and $E (P) \geq 0$.

E(P) is zero for n points while E(A) is zero for 2^n points in characterising function space.

(3) The maximum value of E(P) is 1/n [n log n -(1-n) log (1-n)]

The max. value of E(A) is n log 2.

(4) E(P) and E(A) both are concave continuous permutationally symmetric functions of the arguments.

(5) E(P) is a measure of degree of equality of $p_1, p_2, \ldots\ldots p_n$.

It is maximum when probabilities are maximally equal and it is minimum when probabilities are maximally un equal E(A) does not measure equality of $\mu_A(x_1)$, $\mu_A(x_2), \ldots\ldots$.

It measures the total fuzziness of the n support elements.

7.4 Measures of Directed Divergence :

Among many measures defined for probabilistic directed divergence of distribution $P = (p_1, p_2, \ldots\ldots\ldots p_n)$ from another probability distribution $Q = (q_1, q_2, \ldots\ldots\ldots q_n)$ we take the simplest due to Kulback Liebler.

$$D(P:Q) = \sum_{i=1}^{n} p_i \log p_i / q_i + \sum_{i=1}^{n} (1-p_i) \log (1-p_i / 1-q_i) \quad \ldots \quad (a)$$

And for fuzzy sets A & B, we take

$$D(A:B) = \sum_{i=1}^{n} \mu_A(x_i) \log \{ \mu_A(x_i) / \mu_B(x_i) \}$$
$$+ \sum_{i=1}^{n} \{ 1-\mu_A(x_i) \} \log \{ 1-\mu_A(x_i) / 1-\mu_B(x_i) \},$$
$$\ldots \quad \ldots \quad \ldots \quad (b)$$

(1) D (P:Q) $\geq$ 0, and vanishes iff P = Q.

(2) D (P:Q) is a convex function of both P & Q. Its minimum value is zero.

(3) D (A:B) $\geq$ 0, and vanishes iff A = B.

(4) D (A:B) is a convex function of both $\mu_A(x_i)$ & $\mu_B(x_i)$. Its minimum value need not is zero.

7.5 Measures of Inaccuracy :

For probability distributions P and Q, inaccuracy is

defined by

$$I(P:Q) = -\sum_{i=1}^{n} p_i \log q_i - \sum_{i=1}^{n} (1-p_i) \log (1-q_i) \quad \ldots \quad (a)$$

$$= -\sum_{i=1}^{n} p_i \log p_i + \sum_{i=1}^{n} p_i \log p_i/q_i$$

$$-\sum_{i=1}^{n}(1-p_i) \log (1-p_i) + \sum_{i=1}^{n}(1-p_i) \log (1-p_i / 1-q_i)$$

= Measure of entropy of P + Measure of Directed divergence of P from Q.

For two fuzzy sets A and B, the corresponding measure of inaccuracy is defined by

I (A:B) = Measure of Fuzzy entropy of A + Measure of fuzzy directed divergence of A from B.

I (A:B) is minimum when $\mu_A(x_i) = \mu_B(x_i)$ for all values of i

and the minimum value in that case is the fuzzy entropy of the set A.

7.6 Relation between Fuzzy Entropy and Fuzzy Directed Divergence :-

If U is the set of which every element has the maximum fuzziness value i.e. ½ , then

$$D(A:U) = \sum_{i=1}^{n} \mu_A(x_i) \log \{ \mu_A(x_i) / ½ \}$$

$$+ \sum_{i=1}^{n} \{1-\mu_A(x_i)\} \log \{1-\mu_A(x_i) / (1-½)\},$$

$$= n \log 2 - \Big[-\sum_{i=1}^{n} \mu_A(x_i) \log \mu_A(x_i)$$

$$- \sum_{i=1}^{n} \{1-\mu_A(x_i)\} \log \{1-\mu_A(x_i)\} \Big]$$

$$D[A:U] = n \log 2 - E(A)$$

E (A) is therefore a monotonic decreasing function of the directed divergence of A from the most fuzzy set U and that is why it is a measure of fuzziness of the set A.

7.7 <u>Maximum Fuzziness</u> :

To find maximum fuzziness we maximize

$$-\sum_{i=1}^{n} \mu_A(x_i) \log \{ \mu_A(x_i) \}$$

$$-\sum_{i=1}^{n} \{1-\mu_A(x_i)\} \log \{1-\mu_A(x_i)\},$$

Subject to $\sum_{i=1}^{n} \mu_A(x_i) = \alpha_0$

Emax = n log 2, and this arises when α_0 = n/2 i.e. when each $\mu_A(x_i) = ½$.

i.e. when the set is most fuzzy set.

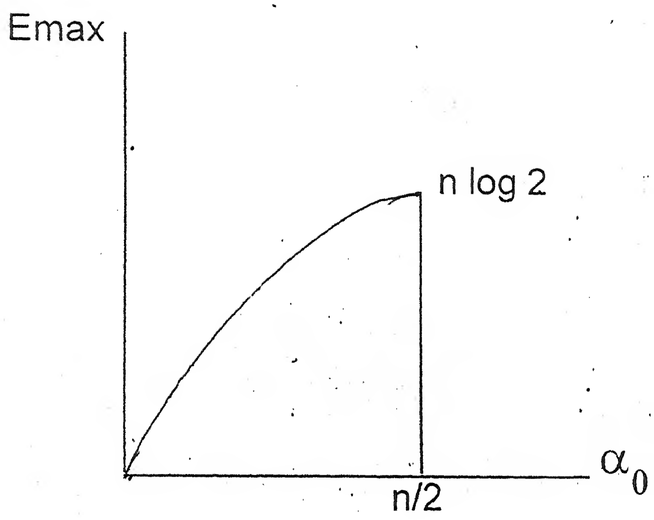

Fig.

7.8 Conclusion :

In above discussion, by quoting one measure of entropy and directed divergence we have cleared the conception of entropy measure and measure of directed divergence. In a similar fashion they can be discussed for other measures of entropy and directed divergence.

References

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A Comparative Study of Fuzzy and Probabilistic Measures of Information

by

Santosh Kumar Singh Bhadauria

Department of Mathematics
Pt. J. N. P. G. College,
Banda (U. P.) – 210 001

Abstract :

Different type of measures such as measures of entropy, measures of directed divergence and measures of symmetric divergence have been defined and discussed for probability distributions in previous years. Recently these measures have also been defined for fuzzy sets in similar manner. But two sets of measures have different interpretations. A comparative study of these two sets of measures has been done in this paper.

Introduction :

Probabilistic measures of information for the probability distribution of the type

$$P = (p_1, p_2, p_3 \ldots\ldots\ldots, p_n) \qquad (1)$$

is defined where

$$0 \leq pi \leq 1 \qquad i = 1, 2, \ldots\ldots, n$$

$$\text{and} \qquad \sum_{i=1}^{n} p_i = 1 \qquad (2)$$

The probabilities are thus dependents and one can not be changed without changing one or more of the others.

Where as fuzzy measures of information are defined for fuzzy sets with fuzzy vectors of the type

$$\{\mu_A(x_1), \mu_A(x_2), \mu_A(x_3), \ldots\ldots\ldots \mu_A(x_n)\} \qquad (3)$$

Where $\mu_A(x_i)$ denotes the degree of membership of the element x_i of the set A..

If $\mu_A(x) = 0$, $x \notin A$

If $\mu_A(x) = 1$, x definitely belongs to A.

$$\text{and } 0 \le \mu_A(x_i) \le 1, \text{ for } i = 1, 2, 3, \ldots\ldots\ldots, n \qquad (4)$$

The sum $\Sigma\mu_A(x_i)$ is not the same for all fuzzy sets. The values of $\mu_A(x_i)$ are independent and any one of them can be changed without affecting the others.

While mathematical forms of two sets of measures seem to be similar, yet they are quite different.

Measures of Entropy :

Among many measures defined for probability entropy and for fuzzy entropy, we take here the simplest measures.

$$E(P) = -\sum_{i=1}^{n} p_i \log p_i - \sum_{i=1}^{n} (1 - p_i)\log(1 - p_i) \qquad (a)$$

and $$E(A) = -\sum_{i=1}^{n} \mu_A(x_i) \log\{\mu_A(x_i)\}$$

$$-\sum_{i=1}^{n} \{1-\mu_A(x_i)\} \log\{1-\mu_A(x_i)\} \qquad \text{(b)}$$

respectively for the purpose of comparison.

(1) E (P) is maximum when $p_1 = p_2 = p_3 = \ldots\ldots = p_n = 1/n$.

E (A) is maximum when

$$\mu_A(x_1) = \mu_A(x_2) = \ldots\ \ldots\ \ldots = \mu_A(x_n) = ½.$$

(2) E (P) is minimum when one of the p_i' is unity and others are zero just as for

$D_1 = (1, 0, 0, \ldots\ldots\ldots, 0)$,

$D_2 = (0, 1, 0, 0, \ldots\ldots\ldots, 0)$,

................................,

$D_n = (0, 0, \ldots\ldots\ldots, 0, 1)$. and $E(P) \geq 0$.

E (P) is zero for n points while E (A) is zero for 2^n points in characterizing function space.

(3) The maximum value of E (P) is $1/n\,[n \log n - (1 - n) \log (1 - n)]$

The max. value of E (A) is n log 2.

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It is maximum when probabilities are maximally equal and it is minimum when probabilities are maximally unequal E (A) does not measure equality of $\mu_A(x_1)$, $\mu_A(x_2)$, ...

It measures the total fuzziness of the n support elements.

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Among many measures defined for probabilistic directed divergence of distribution $P = (p_1, p_2, \ldots\ldots\ldots, p_n)$ from another probability distribution $Q = (q_1, q_2, \ldots\ldots\ldots, q_n)$ we take the simplest due to Kulback Liebler.

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And for fuzzy sets A and B, we take

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(1) $D(P:Q) \geq 0$, and vanishes iff $P = Q$.

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$$\sum_{i=1}^{n} (1-p_i) \log (1-p_i) + \sum_{i=1}^{n} (1-p_i) \log \frac{1-p_i}{1-q_i}$$

= Measure of entropy of P + Measure of Directed divergence of P from Q.

For two fuzzy sets A and B, the corresponding measure of inaccuracy is defined by

I (A : B) = Measure of Fuzzy entropy of A + Measure of fuzzy directed divergence of A from B.

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$$= n \log 2 - \left[-\sum_{i=1}^{n} \mu_A(x_i) \log \mu_A(x_i) \right.$$

$$\left. - \sum_{i=1}^{n} \{1-\mu_A(x_i)\} \log \{1-\mu_A(x_i)\} \right]$$

$$D[A:U] = n \log 2 - E(A)$$

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$$\text{Subject to} \sum_{i=1}^{n} \mu_A(x_i) = \alpha_0$$

$E_{max} = n \log 2$, and this arises when $\alpha_0 = n/2$ i. e.

When each $\mu_A(x_i) = ½$.

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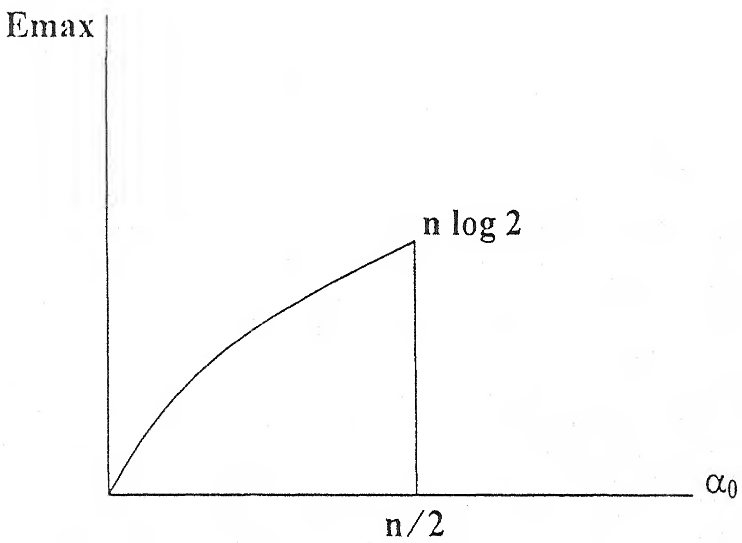

Figure

Conclusion :

In above discussion, by quoting one measure of entropy and directed divergence we have cleared the conception of entropy measure and measure of directed divergence. In a similar fashion they can be discussed for other measures of entropy and directed divergence.

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Fuzzy integration and Fuzzy differentiation

Santosh Kumar Singh Bhadauria

Abstract : On the lines of Lebesgue measures and Lebesgue integrals, fuzzy measure is considered and a definition of fuzzy integral which is a generalization of Lebesgue integral is given. Also in analogy to fuzzy integration, fuzzy differentiation is defined. Differentiation of fuzzy functions is considered. We have considered fuzzy differentiation of a differentiable function.

Fuzzy Integration :

Introduction : One of the first concepts of fuzzy integral was put forward by Sugeno [1972, 1977]; who considered fuzzy measures and suggested a definition of a fuzzy integral which is a generalization of Lebesgue ingtegral : "From the view point of functionals, fuzzy integrals are merely a kind of non linear functionals while Lebesgue integrals are linear one's [Sugeno 1977, p.92]

We shall focus our attention on approaches along the lines of Riemann Integrals. The main references for the following are Dubois and Prade [1980a, 1982b], Aumann [1965] and Nguyen [1978].

The classical concept of integration of a real valued function over a closed interval is generalized in two ways.

(i) The function can be fuzzy function which is to be integrated over a crisp interval, or

(ii) Function is crisp and interval is fuzzy.

Integration of a fuzzy function over a crisp interval : Let the fuzzy function be L.R. Type,

We shall therefore assume that, $\tilde{f} = [f(x), s(x), t(x)]_{LR}$

This is a fuzzy number in LR representation for all $x \in [a, b]$; f, s, t are assumed to be positive integrable functions on $[a, b]$.

Dubois and Prade [1980a, p. 109] have shown that under these conditions

$$\tilde{f}(a, b) = \left(\int_a^b f(x)\,dx, \int_a^b s(x)\,dx, \int_a^b t(x)\,dx\right)_{LR}$$

It is then sufficient to integrate the mean value and spread functions of $\tilde{f}(x)$ over [a,b] and the result will be again LR fuzzy number.

Ex. Let fuzzy function be $\tilde{f}(x) = [f(x), s(x), t(x)]_{LR}$

with the mean function.

$f(x) = x^2$, the spread functions $s(x) = x/4$ and $t(x) = x/2$

$$L(x) = 1/1 + x^2, \quad R(x) = 1/1 + 2|x|$$

To determine the integral from $a = 0$ to $b = 4$; that is to compute $\int_1^4 \tilde{f}$.

According to above formula we compute

$$\int_a^b f(x)\,dx = \int_1^4 x^2\,dx = 21$$

$$\int_a^b s(x)\,dx = \int_1^4 x/4\,dx = 1.875$$

$$\int_a^b t(x)\,dx = \int_1^4 x/2\,dx = 3.75$$

This yields the fuzzy number

$\tilde{f}(a, b) = (21, 1.875, 3.75)_{LR}$ as the value of the fuzzy integral.

One interesting property of fuzzy integrals is as under -

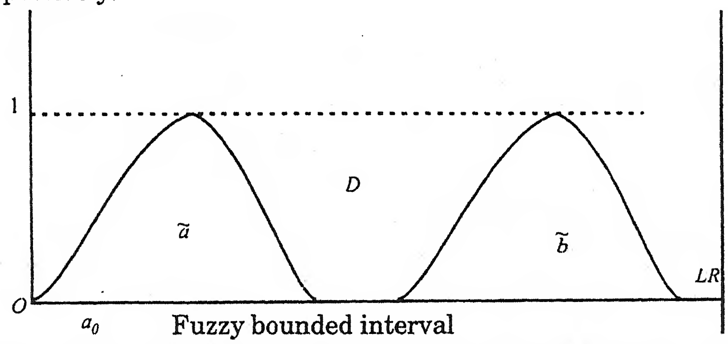
Th. Let $\tilde{f}$ and $\tilde{g}$ be fuzzy functions whose supports are bounded. Then

$$\int_t (\tilde{f} \oplus \tilde{g}) \supseteq \int_t \tilde{f} \oplus \int_t \tilde{g} \qquad \ldots (1)$$

Integration of a (crisp) Real valued function over a fuzzy interval :-

Now, we consider a case for which Dubios and Prade [1982 a.p. 106] proposed a quite interesting solution.

A fuzzy domain of the real line R is assumed to be bounded by two normalized convex fuzzy sets the membership functions of which are $\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{b}}(x)$ respectively.



Fuzzy bounded interval

$\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{b}}(x)$ can be intepretted as degrees of confidence to which x can be taken to be a lower or upper bound of interval.

Definition - Let f be a real-valued function which is integrable in the interval $[a_0, b_0]$, then according to the extension principle the membership function of the integral $j = \int f$ is given by

$$\mu_{\int f} f(z) = \operatorname*{Sup}_{x, y \in j} \{\text{Min}\ \mu_{\tilde{a}}(x), \mu_{\tilde{b}}(x)\}$$

$$Z = \int_x^y f$$

Ex. Let $\tilde{a} = \{(4, .8), (5, 1), (6, 4)\}$

$\tilde{b} = \{(6, .7), (7, 1), (8, 2)\}$

$f(x) = 2, x \in [a_0, b_0] = [4, 8]$

Then $\int f(x)\,dx = \int_{\tilde{a}}^{\tilde{b}} 2x\,dx = 2x \int_{\tilde{a}}^{\tilde{b}}$

The detailed computational results are -

$[a, b]$	$\int_a^b 2\,dx$	min $\{\mu_x(a), \mu_x(b)\}$
[4, 6]	4	.7
[4, 7]	6	.8
[4, 8]	8	.2
[5, 6]	2	.7
[5, 7]	4	1.0
[5, 8]	6	.2
[6, 6]	0	.4
[6, 7]	2	.4
[6, 8]	4	.2

Hence choosing maximum of the membership values for each value of the intergal, yields.

$$\int_f = \{(0, 4), (2, 7), (4, 1), (6, .8), (8, .2)\}$$

Proposition

Let f and g be two functions $f, g : I \to R$, integrable on I

Then $\int_{\tilde{a}}^{\tilde{b}} (f+g) \subseteq \int_{\tilde{a}}^{\tilde{b}} f \oplus \int_{\tilde{a}}^{\tilde{b}} g$

Where $\oplus$ denotes the extended addition

This we illustrate by an example -

Let $f(x) = 2x - 3, g(x) = -2x + 5$

$$\tilde{a} = \{(1, 8), (2, 1), (3, 4)\}$$

$$\tilde{b} = \{(3, 7), (4, 1), (5, 3)\}$$

So $\int_a^b f(x)\,dx = [x^2 - 3x]_a^b$

$$\int_a^b g(x)\,dx = [-x^2 + 5x]_a^b$$

$$\int_a^b \{f(x) + g(x)\}\,dx = [2x]_a^b$$

$[a , b]$	$\int_a^b f(x)\,dx$	min $\{\mu_x(a), \mu_x(b)\}$
[1, 3]	2	.7
[1, 4]	6	.8
[1, 5]	12	.3
[2, 3]	2	.7

[2, 4]	6	1.0
[2, 5]	12	.3
[3, 3]	0	.3
[3, 4]	4	.4
[3, 5]	10	.3

$[a, b]$	$\int_a^b g(x)\,dx$	min $\{\mu_x(a), \mu_x(b)\}$
[1, 3]	2	.7
[1, 4]	0	.8
[1, 5]	− 4	.3
[2, 3]	0	.7
[2, 4]	− 2	1.0
[2, 5]	− 6	.3
[3, 3]	0	.4
[3, 4]	− 2	.4
[3, 5]	− 6	.3

Hence choosing max. value of membership function for each value of the intergral.

$$\int f = \{(0, .4), (2, .7), (4, .4), (6, 1), (10, .3), (12, .3)\}$$

$$\int g = \{(-6, .3), (-4, .3), (-2, 1), (0, .8), (2, .7)\}$$

$[a, b]$	$\int_a^b (f+g)$	min $\{\mu_x(a), \mu_x(b)\}$
[1, 3]	4	.7
[1, 4]	6	.8
[1, 5]	8	.3
[2, 3]	2	.7
[2, 4]	4	1.0
[2, 5]	6	.3
[3, 3]	0	.4
[3, 4]	2	.4
[3, 5]	4	.3

Hence choosing max. value of membership function for each value of the integral.

$$\int (f+g) = \{(0, .4), (2, .7), (4, 1), (6, .8), (8, .3)\}$$

Applying the formula for the extended addition according to the extension principle.

$$\int_{\bar{a}}^{\bar{b}} f \oplus \int_{\bar{a}}^{\bar{b}} g = \{(-6, .3), (-4, .3), (-2, .3), (0, .4), (2, .7), (4, 1),$$

$$(6, .8), (8, .3), (10, .3), (12, .3), (14, .3)\}$$

Therefore we easily verify that

$$\int_{\bar{a}}^{\bar{b}} f \oplus \int_{\bar{a}}^{\bar{b}} g \supseteq \int_{\bar{a}}^{\bar{b}} (f+g)$$

Fuzzy Differentiation :

Introduction : In analogy to fuzzy integration, Fuzzy differentiation is defined.

The results will depend on the type of function considered.

Differentiation of fuzzy functions is considered by Dubois and Prade [1980a, p. 116 and 1982b, p. 227]

Here we consider only fuzzy differentiation of a differentiable function. $f : R \supseteq [a, b] \rightarrow R$ at a "fuzzy point".

"A fuzzy point" $\tilde{X}_0$ [Dubois and Prade 1982b, p. 225] is a convex fuzzy subset of the real line R.

In present case, fuzzy point is considered for which the support is contained in the interval $[a, b]$ that is $S(\tilde{x}) \subseteq [a, b]$.

Such a fuzzy point can be interpreted as the possibility distribution of a point x whose precise location is only approximately known.

The uncertainty of the knowledge about the precise location of the point induces an uncertainty about the derivative $f'(x)$ of a function $f(x)$ at this point. The dervative might be the same for several x belonging to $[a, b]$.

The possibility of $f'(\tilde{X}_0)$ is therefore defined [Zadeh 1978] to be the supremum of the values of the possibilities of $f'(x) = t,\ x \in [a, b]$.

The "Derivative" of a real valued function at a fuzzy point can be interpreted as the fuzzy set $f'(\tilde{x}_0)$ the membership function of which expresses the degree to which a specific $f'(x)$ is, the first derivative of a function f at point $\tilde{X}_0$.

Definition : The membership function of the fuzzy set "derivative of a real valued function at a fuzzy pont $\tilde{X}_0$ " is defined by the extension principle as

$$\mu_{f'(\tilde{x}_0)}(y) = \sup_{x \in f^{-1}(y)} \mu_{\tilde{x}_0}(x).$$

Where $\tilde{X}_0$ is the fuzzy number that characterizes the fuzzy location.

Ex. Let $f(x) = x^3$

and $\tilde{X}_0 = \{(-1, .4), (0, 1), (1, .6)\}$
be a fuzzy location.